Degenerate SDEs in Hilbert Spaces with Rough Drifts *

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Abstract

The existence and uniqueness of mild solutions are proved for a class of degenerate stochastic differential equations on Hilbert spaces where the drift is Dini continuous in the component with noise and Hölder continuous of order larger than $\frac{2}{3}$ in the other component. In the finite-dimensional case the Dini continuity is further weakened. The main results are applied to solve second order stochastic systems driven by space-time white noises.

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Degenerate evolution equation, mild solution, regularization transform.

1 Introduction

Let $\mathbb{H}_i$ $(i = 1, 2, 3)$ be separable Hilbert spaces, and let $\mathscr{L}(\mathbb{H}_i; \mathbb{H}_j)$ be the class of all bounded linear operators from $\mathbb{H}_i$ to $\mathbb{H}_j$ $(1 \le i, j \le 3)$. We shall simply denote the norm and inner product by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ for Hilbert spaces, and let $\|\cdot\|$ stand for the operator norm.

Let W_t be a cylindrical Brownian motion on $\mathbb{H}_3$; i.e. for an orthonormal basis $\{h_i\}_{i\ge 1}$ on $\mathbb{H}_3$, we have

$$W_t = \sum_{i\ge 1} B_t^i h_i,$$

where $\{B_t^i\}_{i\ge 1}$ is a family of independent one-dimensional Brownian motions. Let $\{\mathscr{F}_t\}_{t\ge 0}$ be the natural filtration induced by W_t.

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We consider the following degenerate stochastic evolution equation on $\mathbb{H} := \mathbb{H}_1 \times \mathbb{H}_2$:

$$(1.1) \qquad \begin{cases} \mathrm{d}X_t = \big\{A_1 X_t + BY_t\big\}\mathrm{d}t, \\ \mathrm{d}Y_t = \big\{b_t(X_t, Y_t) + A_2 Y_t\big\}\mathrm{d}t + \sigma_t \mathrm{d}W_t, \end{cases}$$

where $B \in \mathscr{L}(\mathbb{H}_2; \mathbb{H}_1)$, $\sigma : [0,\infty) \to \mathscr{L}(\mathbb{H}_3; \mathbb{H}_2)$, $b : [0,\infty) \times \mathbb{H} \to \mathbb{H}_2$ are measurable and locally bounded, and for every $i = 1, 2$, $(A_i, \mathscr{D}(A_i))$ is a bounded above linear operator generating a strongly continuous semigroup e^{tA_i} on $\mathbb{H}_i$. We will let $\nabla, \nabla^{(1)}$ and $\nabla^{(2)}$ denote the gradient operators on $\mathbb{H}, \mathbb{H}_1$ and $\mathbb{H}_2$ respectively.

Definition 1.1. A continuous adapted process $(X_t, Y_t)_{t \in [0,\zeta)}$ is called a mild solution to (1.1) with life time ζ, if $\zeta > 0$ is an $\mathscr{F}_t$-stopping time such that $\mathbb{P}$-a.s. $\limsup_{t \uparrow \zeta}(|X_t| + |Y_t|) = \infty$ holds on $\{\zeta < \infty\}$ and, $\mathbb{P}$-a.s. for all $t \in [0, \zeta)$,

$$\begin{cases} X_t = \mathrm{e}^{tA_1} X_0 + \displaystyle\int_0^t \mathrm{e}^{(t-s)A_1} BY_s \mathrm{d}s, \\ Y_t = \mathrm{e}^{tA_2} Y_0 + \displaystyle\int_0^t \mathrm{e}^{(t-s)A_2} b_s(X_s, Y_s) \mathrm{d}s + \int_0^t \mathrm{e}^{(t-s)A_2} \sigma_s \mathrm{d}W_s. \end{cases}$$

The purpose of this paper is to investigate the existence/uniqueness of the mild solution under some Dini's type continuity conditions on the drift b. The main idea is to construct a map which transforms the original equation into an equation with regular enough coefficients ensuring the pathwise uniqueness of the solution. This idea goes back to [13, 15] where finite-dimensional SDEs with singular drift are investigated, see also [7, 14] for further developments. In recent years, this argument has been developed in [1, 2, 3, 4, 11] for non-degenerate SDEs in Hilbert spaces. The main difficulty of the study for the present degenerate equation is that the semigroup P_t^0 associated to the linear equation (i.e. $b = 0$) has worse gradient estimates with respect to $x \in \mathbb{H}_1$. More precisely, unlike in the non-degenerate case one has $\|\nabla P_t^0\|_{\infty \to \infty} \approx t^{-1/2}$ for small $t > 0$ which is integrable over $[0, 1]$, for the present model one has $\|\nabla^{(1)} P_t^0\|_{\infty \to \infty} \approx t^{-3/2}$ which is not integrable, where $\|P\|_{\infty \to \infty} := \sup_{\|f\|_\infty \le 1} \|Pf\|_\infty$ for a linear operator P and the uniform norm $\|\cdot\|_\infty$. To reduce the singularity for small $t > 0$, we will use some other norms to replace $\|\cdot\|_{\infty \to \infty}$. This leads to different type continuity conditions on b. Indeed, we will need the Hölder continuity of b in the first component $x \in \mathbb{H}_1$, and a Dini type continuity of b in the second component $y \in \mathbb{H}_2$ as in [11] where the non-degenerate equation is concerned.

To ensure the required gradient estimates on P_t^0, we make the following assumptions on the linear part.

(H1) $\sigma_t \sigma_t^*$ is invertible in $\mathbb{H}_2$ with locally bounded $\|(\sigma_t \sigma_t^*)^{-1}\|$ in $t \ge 0$.

(H2) BB^* is invertible in $\mathbb{H}_1$, and $B\mathrm{e}^{tA_2} = \mathrm{e}^{tA_1}\mathrm{e}^{tA_0}B$ for some $A_0 \in \mathscr{L}(\mathbb{H}_1, \mathbb{H}_1)$ and all $t \ge 0$.

(H3) $-A_2$ is self-adjoint having discrete spectrum $0 < \lambda_1 \le \lambda_2 \le \cdots$ counting multiplicities such that $\sum_{i \ge 1} \frac{1}{\lambda_i^{1-\delta}} < \infty$ for some $\delta \in (0, 1)$.

Since σ_t is locally bounded in $t \geq 0$, B is bounded, and A_1 is bounded above so that $\|e^{A_1 t}\| \leq e^{ct}$ holds for some constant $c \geq 0$, it is well known from [5] that **(H3)** implies the existence, uniqueness and non-explosion of a continuous mild solution to the linear equation, i.e. (1.1) with $b = 0$. As Itô's formula does not apply directly to the mild solution, in the study we will make finite-dimensional approximations. Throughout the paper, we let $\{e_i\}_{i\geq 1}$ be the eigenbasis of A_2, which is an orthonormal basis in $\mathbb{H}_2$ such that $A_2 e_i = -\lambda_i e_i$. For any $n \geq 1$, let $\mathbb{H}_2^{(n)} = \mathrm{span}\{e_1, \cdots, e_n\}$, and let $\pi_2^{(n)} : \mathbb{H}_2 \to \mathbb{H}_2^{(n)}$ be the orthogonal projection. Next, let $\mathbb{H}_1^{(n)} = B\mathbb{H}_2^{(n)}$ and $\pi_1^{(n)} : \mathbb{H}_1 \to \mathbb{H}_1^{(n)}$ be the orthogonal projection. Since BB^* is invertible, we have $\lim_{n\to\infty} \pi_1^{(n)} x = x$ for $x \in \mathbb{H}_1$. Let

$$\pi^{(n)} = (\pi_1^{(n)}, \pi_2^{(n)}) : \mathbb{H} \to \mathbb{H}_n := \mathbb{H}_1^{(n)} \times \mathbb{H}_2^{(n)}.$$

In our study of finite-dimensional approximations, we will need the following assumption which is trivial in the finite-dimensional setting.

(H4) There exists $n_0 \geq 1$ such that for any $n \geq n_0$, $\pi_1^{(n)} B = B\pi_2^{(n)}$ on $\mathbb{H}_2$, and $\pi_1^{(n)} A_1 = A_1 \pi_1^{(n)}$ on $\mathscr{D}(A_1)$.

We introduce the following classes of functions to characterize the continuity modulation of the drift b:

$$\mathscr{D}_0 := \Big\{\phi : [0,\infty) \to [0,\infty) \text{ is increasing with } \phi(0) = 0 \text{ and } \phi(s) > 0 \text{ for } s > 0\Big\},$$
$$\mathscr{D}_1 := \Big\{\phi \in \mathscr{D}_0 : \ \phi^2 \text{ is concave and } \int_0^1 \frac{\phi(s)}{s} \mathrm{d}s < \infty\Big\}.$$

We remark that the condition $\int_0^1 \frac{\phi(s)}{s}\mathrm{d}s < \infty$ is well known as Dini's condition, due to the notion of Dini's continuity. Obviously, the class $\mathscr{D}_1$ contains $\phi(s) := \frac{K}{\{\log(c+s^{-1})\}^{1+r}}$ for constants $K, r > 0$ and large enough $c \geq e$ such that ϕ^2 is concave.

Theorem 1.1. *Assume* **(H1)**-**(H4)**.

(1) *If for any $n \geq 1$ there exist $\alpha_n \in (\frac{2}{3}, 1)$, $\phi_n \in \mathscr{D}_1$ and a constant $K_n > 0$ such that*

$$|b_t(x,y) - b_t(x',y')| \leq K_n|x - x'|^{\alpha_n} + \phi_n(|y - y'|), \quad t \in [0,n], |(x,y)| \vee |(x',y')| \leq n,$$

then for any $(X_0, Y_0) \in \mathbb{H}$, the equation (1.1) has a unique mild solution $(X_t, Y_t)_{t\in[0,\zeta)}$ with life time ζ.

(2) *If moreover*

$$\langle b_t(x, y + y'), y\rangle \leq \ell_t(|x|^2 + |y|^2) + h_t(|y'|), \quad x \in \mathbb{H}_1, y, y' \in \mathbb{H}_2, t \geq 0 \tag{1.2}$$

holds for some increasing function $\ell, h : [0,\infty) \times [0,\infty) \to (0,\infty)$ such that $\int_1^\infty \frac{\mathrm{d}s}{\ell_t(s)} = \infty$ holds for all $t \geq 0$, then the unique mild solution is non-explosive, i.e. the life time $\zeta = \infty$ $\mathbb{P}$-a.s.

To illustrate this result, we consider the following example of second order stochastic system driven by white noise.

Example 1.1. Let $D \subset \mathbb{R}^d$ be a bounded open domain, and let Δ be the Dirichlet Laplace operator on D. Consider the equation

$$
\begin{aligned}
\partial_t^2 u(t,x) =& b_t\big(u(t,x), \partial_t u(t,x) + (-\Delta)^\theta u(t,\cdot)(x)\big) - (-\Delta)^{2\theta} u(t,\cdot)(x) \\
& - 2(-\Delta)^\theta \partial_t u(t,\cdot)(x) + \frac{W(\mathrm{d}t,\mathrm{d}x)}{\mathrm{d}t\mathrm{d}x}, \quad t \ge 0, x \in D.
\end{aligned} \tag{1.3}
$$

Here, $\theta > 0$ is a constant, W is a Brownian sheet (space-time white noise) on $\mathbb{R}^d$, and $b : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is measurable such that for any $T > 0$,

$$
|b_t(u,v) - b_t(u',v')| \le C\big(|u-u'|^\alpha + \phi(|v-v'|)\big), \quad t \in [0,T], u, v, u', v' \in \mathbb{R} \tag{1.4}
$$

holds for some constants $C > 0, \alpha \in (\frac{2}{3}, 1)$, and some $\phi \in \mathscr{D}_1$.

To solve this equation using Theorem 1.1, we take $\mathbb{H}_i = L^2(D;\mathrm{d}x)$ for $i = 1,2,3$, and

$$
W_t = \sum_{i=1}^\infty e_i \int_{[0,t]\times D} e_i(x) W(\mathrm{d}s,\mathrm{d}x)
$$

for $\{e_i\}_{i\ge 1}$ the unitary eigenbasis of Δ. Letting

$$
X_t = u(t,\cdot), \quad Y_t = \partial_t u(t,\cdot) + (-\Delta)^\theta u(t,\cdot),
$$

we reformulate (1.3) as

$$
\begin{cases}
\mathrm{d}X_t = \big\{Y_t - (-\Delta)^\theta X_t\big\}\mathrm{d}t, \\
\mathrm{d}Y_t = \big\{b_t(X_t,Y_t) - (-\Delta)^\theta Y_t\big\}\mathrm{d}t + \mathrm{d}W_t.
\end{cases} \tag{1.5}
$$

Obviously, assumptions **(H1)**, **(H2)** and **(H4)** hold for $B = \sigma_t = I$ (the identity operator) and $A_1 = A_2 = -(-\Delta)^\theta$. Moreover, since the eigenvalues of $(-\Delta)^\theta$ satisfy $\lambda_i \ge c i^{2\theta/d}$ for some constant $c > 0$ and all $i \ge 1$, assumption **(H3)** holds for $A_2 = -(-\Delta)^\theta$ provided $\theta > \frac{d}{2}$. Finally, by Jensen's inequality it is easy to see that (1.4) implies

$$
\|b(f,g) - b(\tilde f,\tilde g)\|_{L^2(D)} \le C\big(\|f-\tilde f\|_{L^2(D)} + \phi(\|g-\tilde g\|_{L^2(D)})\big), \quad f,g,\tilde f,\tilde g \in L^2(D;\mathrm{d}x), t \in [0,T],
$$

where the constant C might be different if the volume of D is not equal to 1. Therefore, by Theorem 1.1, for any $\theta > \frac{d}{2}$ the equation (1.5) has a unique mild solution on $L^2(D;\mathrm{d}x) \times L^2(D;\mathrm{d}x)$ which is non-explosive.

Next, we consider the finite-dimensional case, i.e. consider the following degenerate SDE on $\mathbb{R}^m \times \mathbb{R}^d$:

$$
\begin{cases}
\mathrm{d}X_t = \big\{AX_t + BY_t\big\}\mathrm{d}t, \\
\mathrm{d}Y_t = b_t(X_t,Y_t)\mathrm{d}t + \sigma_t \mathrm{d}W_t,
\end{cases} \tag{1.6}
$$

where A is an $m \times m$-matrix, B is an $m \times d$-matrix, $\sigma : [0,\infty) \to \mathscr{L}(\mathbb{H}_3;\mathbb{R}^d)$ is measurable and locally bounded, and W_t is a cylindrical Brownian motion on $\mathbb{H}_3$. In this case, Theorem 1.1 can be improved by using the following larger class $\mathscr{D}_2$ to replace $\mathscr{D}_1$:

$$\mathscr{D}_2 := \left\{ \phi \in \mathscr{D}_0 : \ \phi^2 \text{ is concave}, \int_0^1 \frac{\mathrm{d}t}{t\big(1+\int_t^1 \frac{\phi(s)}{s}\mathrm{d}s\big)^2} = \infty \right\},$$

which includes $\phi(s) := \frac{K}{\sqrt{\log(c+s^{-1})}}$ for some constant $K>0$ and large enough constant $c>0$.

Theorem 1.2. *Let $\mathbb{H}_1 = \mathbb{R}^m$ and $\mathbb{H}_2 = \mathbb{R}^d$ be finite-dimensional. Assume that BB^* and $\sigma_t\sigma_t^*$ are invertible with $(\sigma_t\sigma_t^*)^{-1}$ locally bounded in $t \ge 0$.*

(1) *If for any $n \ge 1$ there exist $\alpha_n \in (\frac{2}{3}, 1)$, $\phi_n \in \mathscr{D}_2$ and a constant $K_n > 0$ such that*

$$|b_t(x,y) - b_t(x',y')| \le K_n|x-x'|^{\alpha_n} + \phi_n(|y-y'|), \quad t\in[0,n], |(x,y)| \vee |(x',y')| \le n,$$

then for any $(X_0,Y_0)\in\mathbb{R}^{m+d}$, the equation (1.6) has a unique solution $(X_t,Y_t)_{t\in[0,\zeta)}$ with life time ζ.

(2) *If moreover*

$$\langle b_t(x,y), y\rangle \le \ell_t(|x|^2+|y|^2), \quad x\in\mathbb{R}^m, y\in\mathbb{R}^d, t\ge 0 \tag{1.7}$$

holds for some increasing function $\ell, h : [0,\infty)\times[0,\infty) \to (0,\infty)$ such that $\int_1^\infty \frac{\mathrm{d}s}{\ell_t(s)} = \infty$ holds for all $t\ge 0$, then the unique mild solution is non-explosive, i.e. the life time $\zeta = \infty$ $\mathbb{P}$-a.s.

Example 1.2. Consider the following second order stochastic differential equation on $\mathbb{R}^d$:

$$\frac{\mathrm{d}^2 X_t}{\mathrm{d}t^2} = b_t\Big(X_t, \frac{\mathrm{d}X_t}{\mathrm{d}t}\Big) + \sigma \dot{W}_t,$$

where W_t is the d-dimensional Brownian, $\sigma\in\mathbb{R}^d\otimes\mathbb{R}^d$ is invertible, and $b:[0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ is measurable such that for any $T>0$ the condition (1.4) holds for some constants $C>0$, $\alpha\in(\frac{2}{3},1)$ and some function $\phi\in\mathscr{D}_2$. By letting $m=d$ and $Y_t = \frac{\mathrm{d}X_t}{\mathrm{d}t}$, we reformulate this equation as

$$\begin{cases} \mathrm{d}X_t = Y_t\,\mathrm{d}t, \\ \mathrm{d}Y_t = b_t(X_t,Y_t)\mathrm{d}t + \sigma\mathrm{d}W_t. \end{cases} \tag{1.8}$$

According to Theorem 1.2, for any initial point this equation has a unique solution which is non-explosive.

We would like to point out that in the finite-dimensional setting, the pathwise uniqueness for equation (1.6) with Hölder continuous drifts has been investigated in a preprint by

Chaudru de Raynal (http://hal.archives-ouvertes.fr/hal-00702532/document). However, we found some gaps in the proof, for instance, the probabilistic representation of the solution is wrongly used and this is crucial in related calculations. Obviously, in Theorem 1.2 the condition on b along the second component y is much weaker than the Hölder continuity.

The remainder of the paper is organized as follows. In Section 2, we investigate gradient estimates on the semigroup $P^0_{s,t}$ associated to the linear equation (i.e. $b = 0$). These gradient estimates are then used in Section 3 to construct and study the regularization transform. In Section 4 we use the regularization transform to represent the mild solution to (1.1), which enables us to prove Theorems 1.1 and Theorem 1.2 in Section 5.

2 Gradient estimates on $P^0_{s,t}$

For any $s \geq 0$, consider the linear equation

$$\begin{cases} \mathrm{d}X^0_{s,t} = \big\{A_1 X^0_{s,t} + BY^0_{s,t}\big\}\mathrm{d}t, \\ \mathrm{d}Y^0_{s,t} = A_2 Y^0_{s,t}\,\mathrm{d}t + \sigma_t \mathrm{d}W_t, \quad t \geq s. \end{cases} \tag{2.1}$$

By **(H1)**-**(H3)** and Duhamel's formula, the unique solution of this equation starting at $(x, y) \in \mathbb{H}$ at time s is given by:

$$\begin{cases} X^0_{s,t} = \mathrm{e}^{(t-s)A_1}x + \displaystyle\int_s^t \mathrm{e}^{(t-r)A_1} BY^0_{s,r}\mathrm{d}r, \\ Y^0_{s,t} = \mathrm{e}^{(t-s)A_2}y + \displaystyle\int_s^t \mathrm{e}^{(t-r)A_2}\sigma_r \mathrm{d}W_r. \end{cases} \tag{2.2}$$

To indicate the dependence on the initial point, we also denote the solution by $(X^0_{s,t}, Y^0_{s,t})(x, y)$. Let $P^0_{s,t}$ be the Markov operator associated to $(X^0_{s,t}, Y^0_{s,t})$, i.e.

$$P^0_{s,t} f(x, y) = \mathbb{E} f((X^0_{s,t}, Y^0_{s,t})(x, y)), \quad t \geq s \geq 0, (x, y) \in \mathbb{H}, f \in \mathscr{B}_b(\mathbb{H}).$$

By the Markov property, we have $P^0_{s,r}P^0_{r,t} = P^0_{s,t}$ for $0 \leq s \leq r \leq t$.

We first present a Bismut type derivative formula for $P^0_{s,t}$. Let

$$Q_t = \int_0^t s(t-s)\mathrm{e}^{sA_0} BB^* \mathrm{e}^{sA_0^*}\mathrm{d}s, \quad t > 0,$$

where A_0 is in **(H2)**. Since BB^* is invertible and A_0 is bounded, Q_t^{-1} is invertible for every $t > 0$, and for any $T > 0$ there exists a constant $c > 0$ such that

$$\|Q_t^{-1}\| \leq \frac{c}{t^3}, \quad t \in (0, T]. \tag{2.3}$$

Next, for any $s \in [0,T)$ and $v = (v_1, v_2) \in \mathbb{H}$, let

$$\begin{aligned} V^v_{s,T} &:= Q^{-1}_{T-s}\bigg[v_1 + \int_s^T \frac{T-r}{T-s} \mathrm{e}^{(r-s)A_0^*} B v_2 \,\mathrm{d}r\bigg],\\ \Phi^v_{s,T}(r) &:= \mathrm{e}^{(r-s)A_2}\bigg[\frac{v_2}{T-s} + \frac{\mathrm{d}}{\mathrm{d}r}\big\{(r-s)(T-r)B^*\mathrm{e}^{(r-s)A_0^*}\big\}V^v_{s,T}\bigg]. \end{aligned}$$

Theorem 2.1. *For any $s \in [0,T)$, $v = (v_1, v_2) \in \mathbb{H}$, $f \in \mathscr{B}_b(\mathbb{H})$, and $(x,y) \in \mathbb{H}$, there holds*

$$(\nabla_v P^0_{s,T} f)(x,y) = \mathbb{E}\bigg[f\big((X^0_{s,T}, Y^0_{s,T})(x,y)\big) \int_s^T \big\langle \sigma_r^*(\sigma_r\sigma_r^*)^{-1}\Phi^v_{s,T}(r), \mathrm{d}W_r\big\rangle\bigg]. \tag{2.4}$$

Proof. We use the argument of coupling by change of measures as in [6] where the finite-dimensional case is considered. For any $\varepsilon \in [0,1)$, let $(X^\varepsilon_{s,t}, Y^\varepsilon_{s,t})_{t\ge s}$ solve the equation

$$\begin{cases} \mathrm{d}X^\varepsilon_{s,t} = \big\{A_1 X^\varepsilon_{s,t} + BY^\varepsilon_{s,t}\big\}\mathrm{d}t, & X^\varepsilon_{s,s} = x + \varepsilon v_1,\\ \mathrm{d}Y^\varepsilon_{s,t} = \big\{A_2 Y^\varepsilon_{s,t} - \varepsilon \Phi^v_{s,T}(t)\big\}\,\mathrm{d}t + \sigma_t \mathrm{d}W_t, & Y^\varepsilon_{s,s} = y + \varepsilon v_2. \end{cases} \tag{2.5}$$

Noticing that

$$\begin{cases} \mathrm{d}(X^\varepsilon_{s,t} - X^0_{s,t}) = \big\{A_1(X^\varepsilon_{s,t} - X^0_{s,t}) + B(Y^\varepsilon_{s,t} - Y^0_{s,t})\big\}\mathrm{d}t,\\ \mathrm{d}(Y^\varepsilon_{s,t} - Y^0_{s,t}) = \big\{A_2(Y^\varepsilon_{s,t} - Y^0_{s,t}) - \varepsilon\Phi^v_{s,T}(t)\big\}\mathrm{d}t, \end{cases}$$

by Duhamel's formula and the definition of $\Phi^v_{s,T}$, we have

$$\begin{aligned} Y^\varepsilon_{s,t} - Y^0_{s,t} &= \varepsilon \mathrm{e}^{(t-s)A_2} v_2 - \varepsilon \int_s^t \mathrm{e}^{(t-r)A_2}\Phi^v_{s,T}(r)\mathrm{d}r\\ &= \varepsilon \mathrm{e}^{(t-s)A_2}\bigg[\frac{T-t}{T-s} v_2 - (t-s)(T-t)B^*\mathrm{e}^{(t-s)A_0^*}V^v_{s,T}\bigg]. \end{aligned} \tag{2.6}$$

On the other hand, by **(H2)**, we also have

$$\begin{aligned} X^\varepsilon_{s,t} - X^0_{s,t} &= \varepsilon \mathrm{e}^{(t-s)A_1} v_1 + \int_s^t \mathrm{e}^{(t-r)A_1} B(Y^\varepsilon_{s,r} - Y^0_{s,r})\mathrm{d}r\\ &= \varepsilon \mathrm{e}^{(t-s)A_1}\bigg[v_1 + \int_s^t \mathrm{e}^{(r-s)A_0}\Big(\frac{T-r}{T-s}Bv_2 - (r-s)(T-r)BB^*\mathrm{e}^{(r-s)A_0^*}V^v_{s,T}\Big)\mathrm{d}r\bigg]. \end{aligned} \tag{2.7}$$

In particular, by the definition of $V^v_{s,T}$, (2.6) and (2.7) imply

$$(X^\varepsilon_{s,T}, Y^\varepsilon_{s,T}) = (X^0_{s,T}, Y^0_{s,T}), \quad \varepsilon \in (0,1). \tag{2.8}$$

Now, since $\sup_{t\in[s,T]} |\Phi^v_{s,T}(t)| < \infty$, by Girsanov's theorem,

$$W^\varepsilon_t := W_t - \varepsilon \int_s^t \sigma_r^*(\sigma_r\sigma_r^*)^{-1}\Phi^v_{s,T}(r)\mathrm{d}r, \quad t \in [s,T]$$

is a cylindrical Brownian motion on $\mathbb{H}_2$ under the probability measure $\mathrm{d}\mathbb{P}_\varepsilon := R_\varepsilon \mathrm{d}\mathbb{P}$, where

$$R_\varepsilon := \exp\left[\varepsilon \int_s^T \left\langle \sigma_r^*(\sigma_r\sigma_r^*)^{-1}\Phi_{s,T}^v(r), \mathrm{d}W_r\right\rangle - \frac{\varepsilon^2}{2}\int_s^T \left|\sigma_r^*(\sigma_r\sigma_r^*)^{-1}\Phi_{s,T}^v(r)\right|^2 \mathrm{d}r\right]. \tag{2.9}$$

Hence, if we write (2.5) as

$$\begin{cases} \mathrm{d}X_{s,t}^\varepsilon = \left\{A_1 X_{s,t}^\varepsilon + BY_{s,t}^\varepsilon\right\}\mathrm{d}t, & X_{s,s}^\varepsilon = x + \varepsilon v_1, \\ \mathrm{d}Y_{s,t}^\varepsilon = A_2 Y_{s,t}^\varepsilon \,\mathrm{d}t + \sigma_t \mathrm{d}W_t^\varepsilon, & Y_{s,s}^\varepsilon = y + \varepsilon v_2, \end{cases}$$

then by the weak uniqueness of the solution, we obtain

$$P_{s,T}^0 f(x+\varepsilon v_1, y+\varepsilon v_2) = \mathbb{E}_{\mathbb{P}_\varepsilon}\left[f(X_{s,T}^\varepsilon, Y_{s,T}^\varepsilon)\right] = \mathbb{E}\left[R_\varepsilon f(X_{s,T}^\varepsilon, Y_{s,T}^\varepsilon)\right].$$

Combining this with (2.8) and (2.9), we arrive at

$$\begin{aligned}(\nabla_v P_{s,T}^0 f)(x,y) &= \lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\frac{R_\varepsilon - 1}{\varepsilon} f\big((X_{s,T}^0, Y_{s,T}^0)(x,y)\big)\right] \\ &= \mathbb{E}\left[f\big((X_{s,T}^0, Y_{s,T}^0)(x,y)\big)\int_s^T \left\langle \sigma_r^*(\sigma_r\sigma_r^*)^{-1}\Phi_{s,T}^v(r), \mathrm{d}W_r\right\rangle\right].\end{aligned}$$

The proof is finished. □

Remark 2.1. In formula (2.4), although the operator A_1 does not appear explicitly, it is used in (2.7) implicitly though assumption **(H2)**. Of course, in the finite-dimensional case this assumption is not needed, see [6, 14].

We note that the derivative formula in Theorem 2.1 also applies to Hilbert-valued map $f \in \mathscr{B}_b(\mathbb{H};\tilde{\mathbb{H}})$ by expanding f along an orthonormal basis of $\tilde{\mathbb{H}}$, where $\tilde{\mathbb{H}}$ is a separable Hilbert space. Moreover, by the semigroup property, formula (2.4) also implies high order derivative formulas. For instance, for $t \in (s,T)$ and $v, \tilde{v} \in \mathbb{H}$, (2.2) implies

$$v_t := \nabla_v (X_{s,t}^0, Y_{s,t}^0) = \left(\mathrm{e}^{(t-s)A_1}v_1 + \int_s^t \mathrm{e}^{(t-r)A_1}B\mathrm{e}^{(r-s)A_2}v_2\,\mathrm{d}r,\ \mathrm{e}^{(t-s)A_2}v_2\right).$$

Then by $P_{s,T}^0 f = P_{s,t}^0 P_{t,T}^0 f$ and (2.2), (2.4), we have

$$\begin{aligned}\nabla_v\nabla_{\tilde{v}} P_{s,T}^0 f &= \nabla_v \mathbb{E}\left[(P_{t,T}^0 f)(X_{s,t}^0, Y_{s,t}^0)\int_s^t \left\langle \sigma_r^*(\sigma_r\sigma_r^*)^{-1}\Phi_{s,t}^{\tilde{v}}(r), \mathrm{d}W_r\right\rangle\right] \\ &= \mathbb{E}\left[\left(\nabla_v P_{t,T}^0 f(X_{s,t}^0, Y_{s,t}^0)\right)\int_s^t \left\langle \sigma_r^*(\sigma_r\sigma_r^*)^{-1}\Phi_{s,t}^{\tilde{v}}(r), \mathrm{d}W_r\right\rangle\right] \\ &= \mathbb{E}\left[\left(\nabla_{v_t} P_{t,T}^0 f\right)(X_{s,t}^0, Y_{s,t}^0)\int_s^t \left\langle \sigma_r^*(\sigma_r\sigma_r^*)^{-1}\Phi_{s,t}^{\tilde{v}}(r), \mathrm{d}W_r\right\rangle\right] \\ &= \mathbb{E}\Bigg[f(X_{s,T}^0, Y_{s,T}^0)\left(\int_t^T \left\langle \sigma_r^*(\sigma_r\sigma_r^*)^{-1}\Phi_{t,T}^{v_t}(r), \mathrm{d}W_r\right\rangle\right) \\ &\qquad\qquad \times\left(\int_s^t \left\langle \sigma_r^*(\sigma_r\sigma_r^*)^{-1}\Phi_{s,t}^{\tilde{v}}(r), \mathrm{d}W_r\right\rangle\right)\Bigg].\end{aligned} \tag{2.10}$$

We will use (2.4) and (2.10) to estimate derivatives of $P^0_{s,T}f$ for $f \in \mathscr{B}_b(\mathbb{H};\tilde{\mathbb{H}})$ in terms of the norm

$$\|f\|_{\phi,\psi} := \|f\|_\infty + \sup_{(x,y)\neq(x',y')\in\mathbb{H}} \frac{|f(x,y)-f(x',y')|}{\phi(|x-x'|)+\psi(|y-y'|)},$$

where $\phi,\psi \in \mathscr{D}_0$ and $\|\cdot\|_\infty$ is the uniform norm. Let

$$\mathscr{C}_{\phi,\psi}(\mathbb{H};\tilde{\mathbb{H}}) := \Big\{ f \in \mathscr{B}_b(\mathbb{H};\tilde{\mathbb{H}}):\ \|f\|_{\phi,\psi} < \infty \Big\}.$$

Then $(\mathscr{C}_{\phi,\psi}(\mathbb{H};\tilde{\mathbb{H}}), \|\cdot\|_{\phi,\psi})$ is a Banach space. In particular, for any $\alpha \in [0,1]$, if we let $\gamma_\alpha(s) = s^\alpha 1_{(0,\infty)}(s)$, then for $\alpha,\beta \in [0,1]$, $\mathscr{C}_{\gamma_\alpha,\gamma_\beta}(\mathbb{H};\tilde{\mathbb{H}})$ is the usual Hölder space and

$$\|f\|_{\gamma_\alpha,\gamma_\beta} = \|f\|_\infty + \sup_{(x,y)\neq(x',y')\in\mathbb{H}} \frac{|f(x,y)-f(x',y')|}{|x-x'|^\alpha + |y-y'|^\beta}.$$

Note that $\|f\|_{\gamma_0,\gamma_0} \approx \|f\|_\infty$.

Corollary 2.2. *Assume* **(H1)**-**(H3)** *and let $T>0$ be fixed. Let $\nabla^{(i)}$ denote the gradient operator on $\mathbb{H}_i, i=1,2$.*

(1) *There exists a constant $C>0$ such that for any $\alpha \in [0,1]$,*

$$\|\nabla^{(1)} P^0_{s,t} f\|_\infty \le \frac{C\|f\|_{\gamma_\alpha,\gamma_0}}{(t-s)^{\frac{3(1-\alpha)}{2}}},\quad 0 \le s < t \le T, f \in \mathscr{C}_{\gamma_\alpha,\gamma_0}(\mathbb{H};\tilde{\mathbb{H}}).$$

(2) *There exists a constant $C>0$ such that for any $\alpha \in [0,1]$ and $\phi \in \mathscr{D}_0$ with ϕ^2 concave,*

$$\|\nabla^{(2)} P^0_{s,t} f\|_\infty \le \frac{C\|f\|_{\gamma_\alpha,\phi}}{\sqrt{t-s}} \Big[(t-s)^{\frac{\alpha(2+\delta)}{2}} + \phi\big(C(t-s)^{\frac{\delta}{2}}\big)\Big]$$

holds for all $0 \le s < t \le T$ and $f \in \mathscr{C}_{\gamma_\alpha,\phi}(\mathbb{H};\tilde{\mathbb{H}})$, where $\delta \in (0,1)$ is in **(H3)**. *In particular,*

$$\|\nabla^{(2)} P^0_{s,t} f\|_\infty \le \frac{C\|f\|_\infty}{\sqrt{t-s}},\quad 0 \le s < t \le T, f \in \mathscr{B}_b(\mathbb{H};\tilde{\mathbb{H}}).$$

Proof. (1) By the interpolation theorem (cf. [9, Theorem 1.2.1]), it suffices to prove it for $\alpha = 0,1$.

(1a) Let $\alpha = 1$. For any $v_1 \in \mathbb{H}_1$, (2.2) implies

$$\nabla^{(1)}_{v_1} Y^0_{s,t} = 0,\quad \nabla^{(1)}_{v_1} X^0_{s,t} = \mathrm{e}^{(t-s)A_1} v_1. \tag{2.11}$$

So, for any $f \in C^1_b(\mathbb{H};\tilde{\mathbb{H}})$,

$$|\nabla^{(1)}_{v_1} P^0_{s,t} f| = \left|\mathbb{E}\left[\left(\nabla^{(1)}_{\nabla^{(1)}_{v_1} X^0_{s,t}} f\right)(X^0_{s,t}, Y^0_{s,t})\right]\right| \le \|f\|_{1,0} |\mathrm{e}^{(t-s)A_1} v_1| \le \|f\|_{1,0}|v_1|.$$

Thus, assertion (1) is proved for $\alpha = 1$.

(1b) Let $\alpha = 0$ and $0 \le s < t \le T, v \in \mathbb{H}$. By **(H1)** and the definitions of $\Phi^v_{s,t}$ and $V^v_{s,t}$, there exists a constant $C_1 > 0$ such that

$$\int_s^t |\sigma_r^*(\sigma_r\sigma_r^*)^{-1}\Phi^v_{s,t}(r)|^2 \mathrm{d}r \le C_1 \left[\frac{|v_1|^2}{(t-s)^3} + \frac{|v_2|^2}{t-s}\right]. \tag{2.12}$$

Combining this with (2.4) we obtain

$$\begin{aligned} |\nabla_v P^0_{s,t} f|^2 &\le (P^0_{s,t}|f|^2) \int_s^t |\sigma_r^*(\sigma_r\sigma_r^*)^{-1}\Phi^v_{s,t}(r)|^2 \mathrm{d}r \\ &\le C_1(P^0_{s,t}|f|^2) \left[\frac{|v_1|^2}{(t-s)^3} + \frac{|v_2|^2}{t-s}\right]. \end{aligned} \tag{2.13}$$

In particular, with $v_2 = 0$ this implies assertion (1) for $\alpha = 0$.

(2) For $v_2 \in \mathbb{H}_2$ and $(x,y) \in \mathbb{H}$, let $(X^0_{s,t}, Y^0_{s,t}) = (X^0_{s,t}, Y^0_{s,t})(x,y)$ and

$$\tilde{x} = \mathrm{e}^{(t-s)A_1}x + \int_s^t \mathrm{e}^{(t-r)A_1} B \mathrm{e}^{(r-s)A_2} y \mathrm{d}r, \quad \tilde{y} = \mathrm{e}^{(t-s)A_2} y.$$

Moreover, let

$$\xi = \int_s^t \mathrm{e}^{(t-r)A_1} B \mathrm{d}r \int_s^r \mathrm{e}^{(r-r')A_2}\sigma_{r'} \mathrm{d}W_{r'}, \quad \eta = \int_s^t \mathrm{e}^{(t-r)A_2}\sigma_r \mathrm{d}W_r.$$

Since $\mathbb{E}\int_s^t \left\langle \sigma_r^*(\sigma_r\sigma_r^*)^{-1}\Phi^v_{s,t}(r), \mathrm{d}W_r\right\rangle = 0$, applying (2.4) with $v = (0, v_2)$ and using (2.12), we obtain

$$\begin{aligned} |\nabla^{(2)}_{v_2} P^0_{s,t} f|(x,y) &\le \left|\mathbb{E}\left[\Big\{f(X^0_{s,t}, Y^0_{s,t}) - f(\tilde{x},\tilde{y})\Big\} \int_s^t \left\langle \sigma_r^*(\sigma_r\sigma_r^*)^{-1}\Phi^v_{s,t}(r), \mathrm{d}W_r\right\rangle\right]\right| \\ &\le \|f\|_{\gamma_\alpha,\phi}\mathbb{E}\left[\Big(|\xi|^\alpha + \phi(|\eta|)\Big)\left|\int_s^t \left\langle \sigma_r^*(\sigma_r\sigma_r^*)^{-1}\Phi^v_{s,t}(r), \mathrm{d}W_r\right\rangle\right|\right] \\ &\le \frac{C\|f\|_{\gamma_\alpha,\phi}|v_2|}{\sqrt{t-s}}\sqrt{\mathbb{E}(|\xi|^{2\alpha} + \phi(|\eta|))^2}. \end{aligned} \tag{2.14}$$

Noting that **(H3)** implies

$$\begin{aligned} &\int_s^t \|\mathrm{e}^{(t-r)A_2}\sigma_r\|^2_{HS} \mathrm{d}r \le c_1 \sum_{i\ge1} \int_s^t \mathrm{e}^{-2\lambda_i(t-r)} \mathrm{d}r \\ &\le c_1 \sum_{i\ge1} \frac{1-\mathrm{e}^{-2\lambda_i(t-s)}}{2\lambda_i} \le c_1 \sum_{i\ge1} \frac{(2\lambda_i(t-s))^\delta}{2\lambda_i} = c_2(t-s)^\delta, \end{aligned} \tag{2.15}$$

for $c_1 := \sup_{t\in[0,T]} \|\sigma_t\|^2, c_2 := 2^{\delta-1}c_1\sum_{i\ge1}\frac{1}{\lambda_i^{1-\delta}} < \infty$, by Jensen's inequality we have

$$\mathbb{E}|\xi|^{2\alpha} \le c_3(t-s)^{(2+\delta)\alpha}, \quad \mathbb{E}\phi(|\eta|)^2 \le (\phi(\mathbb{E}|\eta|))^2 \le \Big(\phi\big([c_2(t-s)]^{\delta/2}\big)\Big)^2$$

for some constant $c_3 > 0$. Combining this with (2.14) we prove the first assertion in (2), which implies the second assertion by taking $\alpha = 0$ and $\phi = \gamma_0 = 1$. $\square$

Corollary 2.3. *Assume* **(H1)-(H3)**. *For any $T > 0, \alpha \in [0,1]$ and $\phi \in \mathscr{D}_0$ with concave ϕ^2, there exist constants $C_1, C_2, C_3 > 0$ such that for any $f \in \mathscr{B}_b(\mathbb{H}; \tilde{\mathbb{H}})$,*

$$\|\nabla^{(2)}_{\tilde{v}_2}\nabla^{(2)}_{v_2} P^0_{s,t} f\|_\infty \le C_1 \|f\|_{\gamma_\alpha,\phi} |v_2||\tilde{v}_2| \frac{(t-s)^{\alpha(2+\delta)/2} + \phi(C_2(t-s)^{\delta/2})}{t-s},$$

$$\|\nabla^{(1)}_{v_1}\nabla^{(2)}_{v_2} P^0_{s,t} f\|_\infty \le \frac{C_3\|f\|_{\gamma_\alpha,\gamma_0}|v_1||v_2|}{(t-s)^{(4-3\alpha)/2}}, \quad 0 \le s < t \le T, v_1 \in \mathbb{H}_1, v_2, \tilde{v}_2 \in \mathbb{H}_2.$$

Proof. By the second equality in (2.10) for $v = (0, v_2)$ and $\tilde{v} = (0, \tilde{v}_2)$, we obtain

$$\nabla^{(2)}_{\tilde{v}_2}\nabla^{(2)}_{v_2} P^0_{s,t} f = \mathbb{E}\left[\left(\nabla^{(2)}_{\tilde{v}_2} P^0_{\frac{s+t}{2},t} f(X^0_{s,\frac{s+t}{2}}, Y^0_{s,\frac{s+t}{2}})\right)\int_s^{\frac{s+t}{2}} \left\langle \sigma_r^*(\sigma_r\sigma_r^*)^{-1}\Phi^v_{s,\frac{s+t}{2}}(r), \mathrm{d}W_r\right\rangle\right].$$

Combining this with the first inequality in Corollary 2.2 (2) and (2.12), we derive

$$\begin{aligned}\|\nabla^{(2)}_{\tilde{v}_2}\nabla^{(2)}_{v_2} P^0_{s,t} f\|_\infty &\le \|\nabla^{(2)}_{\tilde{v}_2} P^0_{\frac{s+t}{2},t} f\|_\infty \left[\mathbb{E}\int_s^{\frac{s+t}{2}} \left|\sigma_r^*(\sigma_r\sigma_r^*)^{-1}\Phi^v_{s,\frac{s+t}{2},v}(r)\right|^2 \mathrm{d}r\right]^{\frac{1}{2}} \\ &\le C_1\|f\|_{\gamma_\alpha,\phi}|\tilde{v}_2||v_2| \frac{(t-s)^{\alpha(2+\delta)/2} + \phi(C_2(t-s)^{\delta/2})}{t-s}.\end{aligned}$$

Similarly, with $\tilde{v} = (v_1, 0)$ in place of $(0, \tilde{v}_2)$ the second equality in (2.10) implies

$$\nabla^{(1)}_{v_1}\nabla^{(2)}_{v_2} P^0_{s,t} f = \mathbb{E}\left[\left(\nabla^{(1)}_{v_1} P^0_{\frac{s+t}{2},t} f\big(X^0_{s,\frac{s+t}{2}}, Y^0_{s,\frac{s+t}{2}}\big)\right)\int_s^{\frac{s+t}{2}} \left\langle \sigma_r^*(\sigma_r\sigma_r^*)^{-1}\Phi^v_{s,\frac{s+t}{2}}(r), \mathrm{d}W_r\right\rangle\right].$$

By using Corollary 2.2 (1) and (2.12), we prove the second inequality. □

Finally, we apply the above derivative estimates to the resolvent

$$R^\lambda_{s,t} f := \int_s^t \mathrm{e}^{-(r-s)\lambda} P^0_{s,r} f_r \mathrm{d}r, \quad \lambda \ge 0, 0 \le s \le t, f \in \mathscr{B}_b([0,T]\times\mathbb{H}; \tilde{\mathbb{H}}),$$

which will be used in the next section to construct the regularization transform. For any $f \in \mathscr{B}_b([0,T]\times\mathbb{H}; \tilde{\mathbb{H}})$, we simply denote

$$\|f\|_{\phi,\psi} = \sup_{t\in[0,T]} \|f_t\|_{\phi,\psi}, \quad \phi, \psi \in \mathscr{D}_0.$$

Corollary 2.4. *Assume* **(H1)-(H3)** *and let $T > 0$ be fixed.*

1. *$R^\lambda_{s,t} f \in \mathscr{B}_b([0,T]; \mathscr{C}_{\gamma_0,\gamma_1})$ for any $\lambda \ge 0, 0 \le s \le t$ and $f \in \mathscr{B}_b([0,T]\times\mathbb{H}; \tilde{\mathbb{H}})$. Moreover, there exists a constant $C > 0$ such that for any $f \in \mathscr{B}_b([0,T]\times\mathbb{H}; \tilde{\mathbb{H}})$,*

   $$\|\nabla^{(2)} R^\lambda_\cdot f\|_\infty := \sup_{0\le s\le t\le T, z\in\mathbb{H}} \|\nabla^{(2)} R^\lambda_{s,t} f(z)\| \le C\|f\|_\infty \left[\frac{1-\mathrm{e}^{-\lambda T}}{\lambda}\right]^{\frac{1}{2}}, \quad \lambda \ge 0.$$

   Finally, for any $\alpha \in (\frac{2}{3}, 1)$ there exists a function $\theta : [0,\infty) \to (0,\infty)$ with $\theta(\lambda) \downarrow 0$ as $\lambda \uparrow \infty$ such that

   $$\|\nabla^{(1)} R^\lambda_\cdot\|_\infty \le \theta(\lambda)\|f\|_{\gamma_\alpha,\gamma_0}, \quad \lambda \ge 0, f \in \mathscr{B}_b([0,T]; \mathscr{C}_{\gamma_\alpha,\gamma_0}(\mathbb{H}; \tilde{\mathbb{H}})).$$

(2) *For any* $\alpha \in (\frac{2}{3}, 1)$ *and* $\phi \in \mathscr{D}_1$*, there exists* $0 < \theta(\lambda) \downarrow 0$ *as* $\lambda \uparrow \infty$ *such that*

$$\|\nabla\nabla^{(2)} R^\lambda_{s,t} f\|_\infty \le \theta(\lambda)\|f\|_{\gamma_\alpha,\phi}, \quad 0 \le s \le t \le T, f \in \mathscr{B}_b([0,T]; \mathscr{C}_{\gamma_\alpha,\phi}(\mathbb{H}; \tilde{\mathbb{H}})).$$

(3) *Let* $\alpha \in (\frac{2}{3}, 1)$ *and* $\phi \in \mathscr{D}_0$ *with concave* ϕ^2*. Then there exists a constant* $C > 0$ *such that for any* $f \in \mathscr{B}_b([0,T] \times \mathbb{H}; \tilde{\mathbb{H}})$,

$$\|\nabla^{(2)}(R^\lambda_{s,t} f)(z) - \nabla^{(2)}(R^\lambda_{s,t} f)(z')\| \le C\|f\|_{\gamma_\alpha,\phi} \inf_{r\in(0,1)} \left\{ r + |z - z'| \left(1 + \int_{r^\delta}^1 \frac{\phi(s)}{s}\,\mathrm{d}s \right) \right\}$$

holds for all $\lambda \ge 0$ *and* $0 \le s \le t \le T$*, where* $\delta \in (0,1)$ *is in* **(H3)**.

Proof. Assertion (1) follows immediately from the definition of $R^\lambda_{s,t}$ and Corollary 2.2. Next, since $\|\cdot\|_{\gamma_\alpha,\gamma_0} \le c\|\cdot\|_{\gamma_\alpha,\phi}$ holds for some constant $c > 0$, Corollary 2.3 implies

$$\|\nabla\nabla^{(2)} P^0_{s,r} f\|_\infty \le C \left[\frac{1}{(r-s)^{(4-3\alpha)/2}} + \frac{\phi(C(r-s)^{\delta/2})}{r-s} \right] \|f\|_{\gamma_\alpha,\phi}. \tag{2.16}$$

Hence,

$$\|\nabla\nabla^{(2)}(R^\lambda_{s,t} f)\|_\infty \le C\|f\|_{\gamma_\alpha,\phi} \int_s^t \mathrm{e}^{-\lambda(r-s)} \left[\frac{1}{(r-s)^{(4-3\alpha)/2}} + \frac{\phi(C(r-s)^{\delta/2})}{r-s} \right] \mathrm{d}r, \tag{2.17}$$

which implies the assertion (2) due to $\alpha \in (\frac{2}{3}, 1)$ and $\int_0^1 \phi(s)/s\mathrm{d}s < \infty$.

Finally, we prove (3) without assuming $\int_0^1 \phi(s)/s\mathrm{d}s < \infty$. By Corollary 2.2 (2), we have

$$\|\nabla^{(2)} P^0_{s,r} f(z) - \nabla^{(2)} P_{s,r} f(z')\| \le \frac{C\|f\|_{\gamma_\alpha,\phi}}{\sqrt{r-s}}, \quad r > s.$$

On the other hand, by (2.16), we have

$$|\nabla^{(2)} P^0_{s,r} f(z) - \nabla^{(2)} P^0_{s,r} f(z')| \le C|z - z'| \left[\frac{1}{(r-s)^{\frac{4-3\alpha}{2}}} + \frac{\phi(C(r-s)^{\frac{\delta}{2}})}{r-s} \right] \|f\|_{\gamma_\alpha,\phi}. \tag{2.18}$$

Combining these together we obtain

$$\begin{aligned} &\|\nabla^{(2)}(R^\lambda_{s,t} f)(z) - \nabla^{(2)}(R^\lambda_{s,t} f)(z')\| \\ &\le C\|f\|_{\gamma_\alpha,\phi} \int_s^t \mathrm{e}^{-\lambda(r-s)} \left[\frac{|z-z'|}{(r-s)^{\frac{4-3\alpha}{2}}} + \frac{1}{\sqrt{r-s}} \wedge \frac{|z-z'|\phi(C(r-s)^{\frac{\delta}{2}})}{r-s} \right] \mathrm{d}s \end{aligned} \tag{2.19}$$

for some constant $C > 0$. Noting that

$$\begin{aligned} &\int_0^T \mathrm{e}^{-\lambda s} \left[\frac{1}{\sqrt{s}} \wedge \frac{|z-z'|\phi(Cs^{\frac{\delta}{2}})}{s} \right] \mathrm{d}s \\ &\le \int_0^{r^2/C^{2/\delta}} \frac{1}{\sqrt{s}}\mathrm{d}s + |z-z'| \int_{(r^2/C^{2/\delta})\wedge T}^T \frac{\phi(Cs^{\frac{\delta}{2}})}{s}\,\mathrm{d}s \\ &\le C'r + C'|z-z'| \left(1 + \int_{r^\delta}^1 \frac{\phi(s)}{s}\,\mathrm{d}s \right), \quad r \in (0,1) \end{aligned}$$

holds for some constant $C' > 0$, the assertion (3) follows from (2.17). $\square$

3 Regularization transform

Throughout this section, we assume that b is bounded. We aim to construct a map $\Theta : [0,T]\times\mathbb{H} \to \mathbb{H}$ such that Θ_t is a diffeomorphism for every $t \in [0,T]$, and $(\bar{X}_t, \bar{Y}_t) := \Theta_t(X_t, Y_t)$ solves an equation which has pathwise uniqueness provided (X_t, Y_t) solves (1.1). In this way we prove the pathwise uniqueness of (1.1).

The transform will be constructed as

$$\Theta_s(x,y) := (x, y + u_s^\lambda(x,y)), \quad s \in [0,T], (x,y) \in \mathbb{H} \tag{3.1}$$

for large $\lambda > 0$, where u_s^λ solves the integral equation

$$u_s^\lambda = \int_s^T \mathrm{e}^{-\lambda(t-s)} P_{s,t}^0 \Big\{ \nabla_{b_t}^{(2)} u_t^\lambda + b_t \Big\} \mathrm{d}t, \quad s \in [0,T]. \tag{3.2}$$

To ensure that Θ_s is a bijection on $\mathbb{H}$ for every $s \in [0,T]$, we need $\|\nabla^{(2)} u_s^\lambda\|_\infty < 1$. So, we first solve (3.2) and estimate $\nabla^{(2)} u_s^\lambda$.

3.1 Gradient estimates on u_s^λ

Proposition 3.1. *Assume* **(H1)**-**(H3)** *and let* $\alpha \in (\frac{2}{3}, 1), \phi \in \mathscr{D}_0$ *and* $T > 0$.

(1) *For any* $R > 0$ *there exists a constant* $\lambda(R) > 0$ *such that* (3.2) *has a unique solution* $u^\lambda \in C_b([0,T]; \mathscr{C}_{\gamma_0,\gamma_1}(\mathbb{H}; \mathbb{H}_2))$ *provided* $\|b\|_\infty \le R$.

(2) *For any* $R > 0$ *there exist a constant* $\lambda(R) > 0$ *and a positive function* θ *on* $[\lambda(R), \infty)$ *with* $\theta(\lambda) \downarrow 0$ *as* $\lambda \uparrow \infty$ *such that if* $\|b\|_{\gamma_\alpha,\phi} \le R$ *then*

$$\sup_{s\in[0,T]} \|\nabla u_s^\lambda\|_\infty \le \delta(\lambda) \tag{3.3}$$

and

$$\sup_{s\in[0,T]} \|\nabla^{(2)} u_s^\lambda(z) - \nabla^{(2)} u_s^\lambda(z')\| \le C \inf_{r\in(0,1)} \left\{ r + |z-z'| \left(1 + \int_{r^\delta}^1 \frac{\phi(s)}{s} \mathrm{d}s \right) \right\} \tag{3.4}$$

hold for all $\lambda \ge \lambda(R)$ *and* $z, z' \in \mathbb{H}$, *where* $\delta \in (0,1)$ *is in* **(H3)**.

(3) *If* $\phi \in \mathscr{D}_1$, *then for any* $R > 0$,

$$\lim_{\lambda\to\infty} \sup_{t\in[0,T]} \|\nabla\nabla^{(2)} u_t^\lambda\|_\infty = 0$$

holds uniformly for b *with* $\|b\|_{\gamma_\alpha,\phi} \le R$.

Proof. (1) Let

$$\mathscr{H} = C_b([0,T]; \mathscr{C}_{\gamma_0,\gamma_1}(\mathbb{H};\mathbb{H}_2)),$$

which is a Banach space with the norm

$$\|f\|_{\mathscr{H}} := \sup_{t\in[0,T]} \|f_t\|_{\mathscr{C}_{\gamma_0,\gamma_1}} = \sup_{t\in[0,T]} \big(\|f_t\|_{\gamma_0,\gamma_1} + \|f_t\|_\infty\big).$$

For any $f \in \mathscr{H}$, let

$$\Gamma_s^\lambda(f) := R_{s,T}^\lambda(\nabla_b^{(2)} f + b) = \int_s^T \mathrm{e}^{-\lambda(t-s)} P_{s,t}^0\{\nabla_{b_t}^{(2)} f_t + b_t\}\mathrm{d}t, \quad s\in[0,T].$$

By Corollary 2.4 (1), there exists a constant $\lambda(R) > 0$ such that $\|b\|_\infty \le R$ implies

$$\|\Gamma^\lambda(f) - \Gamma^\lambda(g)\|_{\mathscr{H}} \le \frac{1}{2}\|f-g\|_{\mathscr{H}}, \quad f,g\in\mathscr{H}, \lambda\ge\lambda(R). \tag{3.5}$$

Thus, by the fixed-point theorem, for $\lambda \ge \lambda(R)$ and $\|b\|_\infty \le R$ the equation (3.2) has a unique solution $u^\lambda \in \mathscr{H}$. Moreover, taking $g = 0$ in (3.5) and by Corollary 2.4 (1), we obtain

$$\|u^\lambda\|_{\mathscr{H}} \le 2\|\Gamma^\lambda(0)\|_{\mathscr{H}} \le \frac{C\|b\|_\infty}{\sqrt{\lambda}}, \quad \lambda\ge\lambda(R). \tag{3.6}$$

In particular,

$$\|\nabla^{(2)} u^\lambda\|_\infty \le \frac{C_1}{\sqrt{\lambda}}, \quad \lambda\ge\lambda(R). \tag{3.7}$$

(2) All constants C_i mentioned in the proof below are uniform in b with $\|b\|_{\gamma_\alpha,\phi} \le R$ and $\lambda \ge \lambda(R)$. By (2.16) and (3.6), we have

$$\begin{aligned}&\|\nabla^{(2)} P_{s,t}^0\{\nabla_{b_t}^{(2)} u_t^\lambda + b_t\}(z) - \nabla^{(2)} P_{s,t}^0\{\nabla_{b_t}^{(2)} u_t^\lambda + b_t\}(z')\|\\ &\le C_2\|u^\lambda\|_{\mathscr{H}} \frac{|z-z'|}{(t-s)^2} \le \frac{C_3|z-z'|}{(t-s)^2}, \quad 0\le s<t\le T.\end{aligned}$$

Combining these with (3.2) we obtain

$$\|\nabla^{(2)} u^\lambda(z) - \nabla^{(2)} u^\lambda(z')\| \le (C_1\vee C_3)\int_0^T \Big(1\wedge \frac{|z-z'|}{t^2}\Big)\mathrm{d}t \le C_4\sqrt{|z-z'|}.$$

Since $\|b\|_{\gamma_\alpha,\gamma_0} \le \|b\|_{\gamma_\alpha,\phi} \le R$, this implies

$$\|\nabla_b^{(2)} u^\lambda + b\|_{\gamma_{1/2},\gamma_0} \le C_5. \tag{3.8}$$

Combining this with (2.16) for $\alpha = \frac{1}{2}$, we obtain

$$\|\nabla^{(2)} P_{s,t}^0\{\nabla_{b_t} u_t^\lambda + b_t\}(z) - \nabla^{(2)} P_{s,t}^0\{\nabla_{b_t} u_t^\lambda + b_t\}(z')\| \le \frac{C_6|z-z'|}{(t-s)^{5/4}}, \quad 0\le s<t\le T.$$

This together with (3.7) leads

$$\|\nabla^{(2)}u^\lambda(z)-\nabla^{(2)}u^\lambda(z')\|\le (C_1\vee C_6)\int_0^T\Big(1\wedge\frac{|z-z'|}{t^{5/4}}\Big)\mathrm{d}t\le C_7|z-z'|^{4/5}.$$

Let $\psi(s)=\sqrt{\phi(s)^2+s}$ such that $\psi\in\mathscr{D}_0$. Since $\|b\|_{\gamma_\alpha,\psi}\le\|b\|_{\gamma_\alpha,\phi}\le R$, this implies

$$\|\nabla_b^{(2)}u^\lambda+b\|_{\gamma_{\alpha\wedge\frac{4}{5}},\psi}\le C_8.$$

Thus, by Corollary 2.4 (3) for $(\alpha\wedge\frac{4}{5},\psi)$ in place of (α,ϕ), we obtain

$$\begin{aligned}\|\nabla^{(2)}u_s^\lambda(z)-\nabla^{(2)}u_s^\lambda(z')\|&\le C_9\inf_{r\in(0,1)}\bigg\{r+|z-z'|\bigg(1+\int_{r^\delta}^1\frac{\psi(s)}{s}\,\mathrm{d}s\bigg)\bigg\}\\&\le C_{10}\inf_{r\in(0,1)}\bigg\{r+|z-z'|\bigg(1+\int_{r^\delta}^1\frac{\phi(s)}{s}\,\mathrm{d}s\bigg)\bigg\}.\end{aligned}$$

Therefore, (3.4) holds for some constant $C>0$. Finally, by (3.2), (3.8) and the last inequality in Corollary 2.4 (1), we prove (3.3) for some positive function δ with $\delta(\lambda)\downarrow 0$ as $\lambda\uparrow\infty$. So, the proof of assertion (2) is finished.

(3) Finally, if $\|b\|_{\gamma_\alpha,\phi}\le R$ for some $\alpha\in(\frac{2}{3},1)$ and $\phi\in\mathscr{D}_1$, then by assertion (2), $\nabla^{(2)}u_s^\lambda$ is Lipschitz continuous uniformly in $s\in[0,T]$ so that

$$\|\nabla_b^{(2)}u^\lambda+b\|_{\gamma_\alpha,\phi}\le C_{11}.$$

So, by Corollary 2.4 (2) and (3.2), we prove assertion (3). □

According to (3.6) we have $\|\nabla^{(2)}u^\lambda\|_\infty<1$ for large $\lambda>0$, so that Θ_s given in (3.1) is a bijection for every $s\in[0,T]$. To figure out the equation satisfied by $(\bar X_t,\bar Y_t):=\Theta_t(X_t,Y_t)$, we need to apply Itô's formula to $u_t^\lambda(X_t,Y_t)$, which is however not available in the infinite-dimensional setting. Thus, we need to investigate approximations on u^λ such that Itô's formula can be established for $u_t^\lambda(X_t,Y_t)$ by finite-dimensional approximations.

3.2 Approximations on u^λ

Let $\mathbb{H}_i^{(n)}(i=1,2),\mathbb{H}^{(n)}$ and $\pi^{(n)}=(\pi_1^{(n)},\pi_2^{(n)})$ be defined in Section 1. Since BB^* is invertible, for any $x\in\mathbb{H}_1$ we have $x=By'$ for $y'=B^*(BB^*)^{-1}x\in\mathbb{H}_2$. So,

$$\lim_{n\to\infty}|\pi^{(n)}(x,y)-(x,y)|\le\lim_{n\to\infty}\big(|\pi_2^{(n)}y-y|+|B(\pi_2^{(n)}y'-y')|\big)=0,\quad (x,y)\in\mathbb{H}. \tag{3.9}$$

Let

$$b_s^{(n)}=\pi_2^{(n)}b_s,\quad \sigma_s^{(n)}=\pi_2^{(n)}\sigma_s,\quad s\ge 0.$$

Since $\mathbb{H}_2^{(n)}$ is A_2-invariant, $\mathbb{H}_1^{(n)}=B\mathbb{H}_2^{(n)}$ and $A_1B=BA_2$, we have

$$\pi_2^{(n)}A_2\pi_2^{(n)}=A_2\pi_2^{(n)},\quad \pi_1^{(n)}A_1\pi_1^{(n)}=A_1\pi_1^{(n)}. \tag{3.10}$$

For any $n \geq 1$, consider the equation

$$u_s^{\lambda,n} = \int_s^T \mathrm{e}^{-\lambda(t-s)} P_{s,t}^0 \Big\{ \nabla^{(2)}_{b_t^{(n)}} u_t^{\lambda,n} + b_t^{(n)} \Big\} \mathrm{d}t, \quad s \in [0,T]. \tag{3.11}$$

Proposition 3.2. *Assume* **(H1)**-**(H4)** *and let* n_0 *be from* **(H4)** *and* $b \in \mathscr{B}_b([0,T]; C_b(\mathbb{H};\mathbb{H}_2))$. *Then there exists a constant* $\lambda_0 > 0$ *such that for any* $\lambda \geq \lambda_0$ *and* $n \geq n_0$, *equation* (3.11) *has a unique solution* $u^{\lambda,n} \in C_b([0,T]; \mathscr{C}_{\gamma_0,\gamma_1}(\mathbb{H};\mathbb{H}_2^{(n)}))$ *with*

$$\lim_{\lambda\to\infty} \sup_{n\geq n_0} \big(\|\nabla^{(2)} u^{\lambda,n}\|_\infty + \|u^{\lambda,n}\|_\infty \big) = 0. \tag{3.12}$$

Moreover, for all $s \in [0,T]$ *and* $z \in \mathbb{H}$, *we have*

$$\lim_{n\to\infty} \big(|u_s^{\lambda,n}(z) - u_s^\lambda(z)| + \|\nabla^{(2)}(u_s^{\lambda,n} - u_s^\lambda)(z)\| \big) = 0, \ \ \lambda \geq \lambda_0. \tag{3.13}$$

Proof. By Proposition 3.1 (1), there exists $\lambda_0 > 0$ such that for any $n \geq 1$ and $\lambda \geq \lambda_0$, the equation (3.11) has a unique solution $u^{\lambda,n} \in C_b([0,T]; \mathscr{C}_{\gamma_0,\gamma_1}(\mathbb{H};\mathbb{H}_2))$. By **(H4)** and (3.10) we have

$$\pi_2^{(n)} P_{s,t}^0 f = P_{s,t}^0 \pi_2^{(n)} f, \quad n \geq n_0, f \in \mathscr{B}_b(\mathbb{H};\mathbb{H}_2).$$

So, it is easy to see that $\pi_2^{(n)} u^{\lambda,n}$ also solves the equation (3.11). By the uniqueness we get $u^{\lambda,n} \in C_b([0,T]; \mathscr{C}_{\gamma_0,\gamma_1}(\mathbb{H};\mathbb{H}_2^{(n)}))$. Obviously, (3.12) follows from (3.6). Let us prove (3.13). By (3.2) and (3.11), we have

$$\begin{aligned} u_s^\lambda - u_s^{\lambda,n} = & \int_s^T \mathrm{e}^{-\lambda(t-s)} P_{s,t}^0 \Big\{ \nabla^{(2)}_{b_t^{(n)}} (u_t^\lambda - u_t^{\lambda,n}) \Big\} \mathrm{d}t \\ & + \int_s^T \mathrm{e}^{-\lambda(t-s)} P_{s,t}^0 \Big\{ \nabla^{(2)}_{b_t - b_t^{(n)}} u_t^\lambda + b_t - b_t^{(n)} \Big\} \mathrm{d}t. \end{aligned} \tag{3.14}$$

Let

$$g_s(z) = \limsup_{n\to\infty} \|\nabla^{(2)}(u_s^\lambda - u_s^{\lambda,n})(z)\|, \quad s \in [0,T], z \in \mathbb{H}.$$

By (3.12), $g \in \mathscr{B}_b([0,T]\times\mathbb{H})$. Moreover, since

$$\|\nabla^{(2)} u^\lambda\|_\infty < \infty, \quad \|b^{(n)}\|_\infty \leq \|b\|_\infty < \infty, \quad \lim_{n\to\infty} |b_t(z) - b_t^{(n)}(z)| = 0,$$

it follows from (2.13) with $v_1 = 0$ and the dominated convergence theorem that

$$\begin{aligned} & \limsup_{n\to\infty} \int_s^T \Big\| \nabla^{(2)} P_{s,t}^0 \Big\{ \nabla^{(2)}_{b_t - b_t^{(n)}} u_t^\lambda + b_t - b_t^{(n)} \Big\}(z) \Big\| \mathrm{d}t \\ & \leq C \limsup_{n\to\infty} \int_s^T \frac{1}{\sqrt{t-s}} \Big(P_{s,t}^0 \big| \nabla^{(2)}_{b_t - b_t^{(n)}} u_t^\lambda + b_t - b_t^{(n)} \big|^2 (z) \Big)^{\frac{1}{2}} \mathrm{d}t = 0. \end{aligned}$$

Combining this with (3.14) and (2.13), we obtain

$$
\begin{aligned}
g_s(z) &\le C \limsup_{n\to\infty} \int_s^T \frac{\mathrm{e}^{-\lambda(t-s)}}{\sqrt{t-s}} \Big(P^0_{s,t}\big|\nabla^{(2)}_{b_t^{(n)}}(u_t^\lambda - u_t^{\lambda,n})\big|^2(z)\Big)^{\frac12}\mathrm{d}t \\
&\le C\|b\|_\infty \int_s^T \frac{\mathrm{e}^{-\lambda(t-s)}}{\sqrt{t-s}} (P^0_{s,t}|g|^2)^{\frac12}(z)\mathrm{d}t \le \frac{C'}{\sqrt{\lambda}}\|g\|_\infty, \quad s\in[0,T], z\in\mathbb{H}, \lambda\ge\lambda_0
\end{aligned}
$$

for some constants $C, C' > 0$ independent of s, z. This implies $g_s(z) = 0$ when λ is large enough. □

4 Representation of Y_t using u^λ

Assume **(H1)-(H4)** and let $b \in \mathscr{B}_b([0,T]; C_b(\mathbb{H};\mathbb{H}_2))$ for some $T > 0$. Let u^λ be constructed in Section 3 for $\lambda \ge \lambda_0$, where λ_0 is in Proposition 3.2.

Theorem 4.1. *Assume* **(H1)-(H4)** *and let* $b \in \mathscr{B}_b([0,T]; C_b(\mathbb{H};\mathbb{H}_2))$ *for some* $T > 0$. *If* $(Z_t)_{t\in[0,T\wedge\tau]} := (X_t, Y_t)_{t\in[0,T\wedge\tau]}$ *solves* (1.1) *for some stopping time* τ, *i.e.* $\mathbb{P}$-*a.s.*

$$
\text{(4.1)}\qquad \begin{cases} X_t = \mathrm{e}^{tA_1}X_0 + \displaystyle\int_0^t \mathrm{e}^{(t-s)A_1} BY_s\mathrm{d}s, \ 0\le t\le T\wedge\tau, \\ Y_t = \mathrm{e}^{tA_2}Y_0 + \displaystyle\int_0^t \mathrm{e}^{(t-s)A_2}\big\{b_s(X_s,Y_s)\mathrm{d}s + \sigma_s\mathrm{d}W_s\big\}, \end{cases}
$$

then for any $\lambda \ge \lambda_0$, $\mathbb{P}$-*a.s.*

$$
\text{(4.2)}\qquad \begin{aligned} Y_t = \mathrm{e}^{tA_2}Y_0 + \mathrm{e}^{tA_2}u_0^\lambda(Z_0) - u_t^\lambda(Z_t) + \int_0^t (\lambda - A_2)\mathrm{e}^{(t-s)A_2}u_s^\lambda(Z_s)\mathrm{d}s \\ + \int_0^t \mathrm{e}^{(t-s)A_2}\Big\{\sigma_s\mathrm{d}W_s + \big(\nabla^{(2)}_{\sigma_s\mathrm{d}W_s}u_s^\lambda\big)(Z_s)\Big\}, \quad 0\le t\le T\wedge\tau. \end{aligned}
$$

Proof. Let $\big(X^0_{s,t}, Y^0_{s,t}\big)_{t\ge s}$ solve the linear equation (2.1) with initial value $z \in \mathbb{H}$. Write

$$
Z^0_{s,t} := \big(X^0_{s,t}, Y^0_{s,t}\big), \quad Z^{(n)}_{s,t} := \pi^{(n)}Z^0_{s,t}.
$$

By **(H4)** and (3.10), $Z^{(n)}_{s,t} = \big(X^{(n)}_{s,t}, Y^{(n)}_{s,t}\big)$ solves the following equation:

$$
\text{(4.3)}\qquad \begin{cases} \mathrm{d}X^{(n)}_{s,t} = \big\{A_1X^{(n)}_{s,t} + BY^{(n)}_{s,t}\big\}\mathrm{d}t, \\ \mathrm{d}Y^{(n)}_{s,t} = A_2Y^{(n)}_{s,t}\mathrm{d}t + \sigma_t^{(n)}\mathrm{d}W_t. \end{cases}
$$

Since $Z^{(n)}_{s,s}(z) = \pi_n z$, by the uniqueness we have

$$
\text{(4.4)}\qquad Z^{(n)}_{s,t}(z) = Z^{(n)}_{s,t}(\pi^{(n)}z).
$$

Let u^λ and $u^{\lambda,n}$ be in Proposition 3.2. Let

$$f_t^{(n)} = \nabla^{(2)}_{b_t^{(n)}} u_t^{\lambda,n} + b_t^{(n)}, \quad t \in [0,T], n \geq n_0.$$

For fixed $\varepsilon_0 > 0$, let

$$F_{s,r}^{(n)}(z) := P^0_{s,r+\varepsilon_0}(f_r^{(n)} \circ \pi^{(n)})(z), \quad 0 \leq s \leq r \leq T, z \in \mathbb{H}, n \geq n_0.$$

By (4.4) we have

$$F_{s,r}^{(n)}(z) = \mathbb{E}\Big[f_r^{(n)}\big(Z^{(n)}_{s,r+\varepsilon_0}(z)\big)\Big] = \mathbb{E}\Big[f_r^{(n)}\big(Z^{(n)}_{s,r+\varepsilon_0}(\pi^{(n)}z)\big)\Big] = F_{s,r}^{(n)}(\pi^{(n)}(z)). \tag{4.5}$$

Since $\varepsilon_0 > 0$ is fixed, by (2.10) we have $F_{s,r}^{(n)} \in C_b^2(\mathbb{H}^{(n)}; \mathbb{H}_2^{(n)})$ with

$$\sup_{0\leq s\leq r\leq T, n\geq n_0} \Big(\|F_{s,r}^{(n)}\|_\infty + \|\nabla F_{s,r}^{(n)}\|_\infty + \|\nabla\nabla F_{s,r}^{(n)}\|_\infty\Big) < \infty. \tag{4.6}$$

By (4.5), (4.3) and Itô's formula, we have

$$\mathrm{d}F_{s,r}^{(n)}(Z_{s,t}^{(n)}) = (\mathscr{L}_t^{(n)} F_{s,r}^{(n)})(Z_{s,t}^{(n)})\mathrm{d}t + \Big(\nabla^{(2)}_{\sigma_t^{(n)}\mathrm{d}W_t} F_{s,r}^{(n)}\Big)(Z_{s,t}^{(n)}), \tag{4.7}$$

where for $F \in C_b^2(\mathbb{H}; \mathbb{H}_2)$ with $F = F \circ \pi^{(n)}$,

$$(\mathscr{L}_t^{(n)} F)(x,y) := \frac{1}{2}\sum_{i,j=1}^n \langle \sigma_t\sigma_t^* e_i, e_j\rangle \nabla^{(2)}_{e_i}\nabla^{(2)}_{e_j} F(x,y) + \nabla^{(1)}_{A_1x+By} F(x,y), \quad (x,y) \in \mathbb{H}.$$

Combining (4.5), (4.6) and (4.7), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} F_{s,r}^{(n)} &= -\lim_{\varepsilon\downarrow 0} \frac{F_{s-\varepsilon,r}^{(n)} - F_{s,r}^{(n)}}{\varepsilon} = -\lim_{\varepsilon\downarrow 0} \frac{\mathbb{E}[F_{s,r}^{(n)}(Z_{s-\varepsilon,s}^{(n)}) - F_{s,r}^{(n)}(Z_{s-\varepsilon,s-\varepsilon}^{(n)})]}{\varepsilon} \\ &= -\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_{s-\varepsilon}^s \mathscr{L}_{s-\varepsilon}^{(n)} F_{s,r}^{(n)}(Z_{s-\varepsilon,t}^{(n)})\mathrm{d}t = -\mathscr{L}_s^{(n)} F_{s,r}^{(n)}, \quad \text{a.e. } s \in (0, \delta + r).\end{aligned} \tag{4.8}$$

Thus, if we write

$$u_{s,\varepsilon_0}^{\lambda,n} := \int_s^T \mathrm{e}^{-\lambda(t-s)} P^0_{s,t+\varepsilon_0}(f_t^{(n)} \circ \pi^{(n)})\mathrm{d}t = \int_s^T \mathrm{e}^{-\lambda(t-s)} F_{s,t}^{(n)}\mathrm{d}t,$$

then $u_{s,\varepsilon_0}^{\lambda,n}$ satisfies

$$\partial_s u_{s,\varepsilon_0}^{\lambda,n} = (\lambda - \mathscr{L}_s^{(n)})u_{s,\varepsilon_0}^{\lambda,n} - P^0_{s,s+\varepsilon_0}\big(\big\{\nabla^{(2)}_{b_s^{(n)}} u_s^{\lambda,n} + b_s^{(n)}\big\} \circ \pi^{(n)}\big). \tag{4.9}$$

Noticing that by (4.5) and definitions,

$$u_{s,\varepsilon_0}^{\lambda,n} = u_{s,\varepsilon_0}^{\lambda,n} \circ \pi^{(n)},$$

we may apply Itô's formula to $u^{\lambda,n}_{s,\varepsilon_0}(Z_s) = u^{\lambda,n}_{s,\varepsilon_0}(\pi^{(n)}Z_s)$ so that (4.1) and (4.9) imply

$$\begin{aligned}
&\mathrm{d}u^{\lambda,n}_{s,\varepsilon_0}(Z_s) = \big(\nabla^{(2)}_{\sigma_s \mathrm{d}W_s}\big)u^{\lambda,n}_{s,\varepsilon_0}(Z_s) + \big(\nabla^{(2)}_{b^{(n)}_s} u^{\lambda,n}_{s,\varepsilon_0} + \mathscr{L}^{(n)}_s u^{\lambda,n}_{s,\varepsilon_0} + \partial_s u^{\lambda,n}_{s,\varepsilon_0}\big)(Z_s)\mathrm{d}s\\
&= \big(\nabla^{(2)}_{\sigma_s \mathrm{d}W_s}\big)u^{\lambda,n}_{s,\varepsilon_0}(Z_s) + \Big[\nabla^{(2)}_{b^{(n)}_s} u^{\lambda,n}_{s,\varepsilon_0} - P^0_{s,s+\varepsilon_0}\big(\big\{\nabla^{(2)}_{b^{(n)}_s} u^{\lambda,n}_s + b^{(n)}_s\big\}\circ\pi^{(n)}\big) + \lambda u^{\lambda,n}_{s,\varepsilon_0}\Big](Z_s)\mathrm{d}s.
\end{aligned}$$

Thus, for any $t \ge 0$, we have

$$\begin{aligned}
&u^{\lambda,n}_{t,\varepsilon_0}(Z_t) - \mathrm{e}^{tA_2}u^{\lambda,n}_{0,\varepsilon_0}(Z_0) = \int_0^t \mathrm{d}(\mathrm{e}^{(t-s)A_2}u^{\lambda,n}_{s,\varepsilon_0}(Z_s))/\mathrm{d}s\\
&= -\int_0^t A_2\mathrm{e}^{(t-s)A_2}u^{\lambda,n}_{s,\varepsilon_0}(Z_s)\mathrm{d}s + \int_0^t \mathrm{e}^{(t-s)A_2}\big(\nabla^{(2)}_{\sigma_s\mathrm{d}W_s}u^{\lambda,n}_{s,\varepsilon_0}\big)(Z_s)\\
&\quad + \int_0^t \mathrm{e}^{(t-s)A_2}\Big(\nabla^{(2)}_{b^{(n)}_s}u^{\lambda,n}_{s,\varepsilon_0} - P^0_{s,s+\varepsilon_0}\big(\nabla^{(2)}_{b^{(n)}_s}u^{\lambda,n}_s\big)\Big)(\pi^{(n)}(Z_s))\mathrm{d}s\\
&\quad + \int_0^t \mathrm{e}^{(t-s)A_2}\Big(\lambda u^{\lambda,n}_{s,\varepsilon_0} - P^0_{s,s+\varepsilon_0}b^{(n)}_s\Big)(\pi^{(n)}(Z_s))\mathrm{d}s.
\end{aligned}\tag{4.10}$$

We claim that the desired formula (4.2) follows by first letting $\varepsilon_0 \downarrow 0$ and then $n \uparrow \infty$. To see this, we write

$$u^{\lambda,n}_{s,\varepsilon_0} - u^{\lambda,n}_s = \int_s^T \mathrm{e}^{-\lambda(t-s)}P^0_{s,t}\big\{P^0_{t,t+\varepsilon_0}(f^{(n)}_t\circ\pi^{(n)}) - f^{(n)}_t\big\}\mathrm{d}t.\tag{4.11}$$

By the boundedness and continuity of $b^{(n)}_s$ and $f^{(n)}_t$, and noting that $\lim_{\varepsilon_0\downarrow 0} P^0_{t,t+\varepsilon_0}f = f$ for $f \in C_b(\mathbb{H};\mathbb{H}_2)$, we obtain

$$\lim_{n\to\infty}\lim_{\varepsilon_0\downarrow 0}(u^{\lambda,n}_{s,\varepsilon_0} - u^{\lambda,n}_s) = 0,\quad \lim_{n\to\infty}\lim_{\varepsilon_0\downarrow 0}(P^0_{s,s+\varepsilon_0}b^{(n)}_s)(\pi^{(n)}(Z_s)) = b_s(Z_s).\tag{4.12}$$

Moreover, (2.13) with $v_1 = 0$ and (4.11) imply

$$\lim_{n\to\infty}\lim_{\varepsilon_0\downarrow 0}\|\nabla^{(2)}(u^{\lambda,n}_{s,\varepsilon_0} - u^{\lambda,n}_s)\| \le \lim_{n\to\infty}\lim_{\varepsilon_0\downarrow 0}\int_s^T \frac{C}{\sqrt{t-s}}\sqrt{P^0_{s,t}|P^0_{t,t+\varepsilon_0}(f^{(n)}_t\circ\pi^{(n)}) - f^{(n)}_t|^2}\,\mathrm{d}t = 0.$$

Combining this with (4.12) and Proposition 3.2, we obtain

$$\lim_{n\to\infty}\lim_{\varepsilon_0\downarrow 0}\Big(|u^{\lambda,n}_{s,\varepsilon_0} - u^\lambda_s| + \|\nabla^{(2)}(u^{\lambda,n}_{s,\varepsilon_0} - u^\lambda_s)\|\Big) = 0.\tag{4.13}$$

Thus, taking inner product for both sides of (4.10) with every e_i and letting $\varepsilon_0 \downarrow 0$ and $n \uparrow \infty$, we conclude that $\mathbb{P}$-a.s. for all $t \in [0, T\wedge\tau]$ and all $i \ge 1$,

$$\begin{aligned}
&\langle u^\lambda_t(Z_t), e_i\rangle - \mathrm{e}^{-\lambda_i t}\langle u^\lambda_0(Z_0), e_i\rangle = \int_0^t \mathrm{e}^{-\lambda_i(t-s)}\lambda_i\langle u^\lambda_s(Z_s), e_i\rangle\mathrm{d}s\\
&+ \int_0^t \mathrm{e}^{-\lambda_i(t-s)}\big\langle(\nabla^{(2)}_{\sigma_s\mathrm{d}W_s}u^\lambda_s)(Z_s), e_i\big\rangle + \int_0^t \mathrm{e}^{-\lambda_i(t-s)}\langle\lambda u^\lambda_s(Z_s) - b_s(Z_s), e_i\rangle\mathrm{d}s.
\end{aligned}$$

That is, $\mathbb{P}$-a.s. for all $t \in [0, T\wedge\tau]$,

$$
\begin{aligned}
u_t^\lambda(Z_t) - \mathrm{e}^{tA_2}u_0^\lambda(Z_0) &= \int_0^t A_2\mathrm{e}^{(t-s)A_2}u_s^\lambda(Z_s)\mathrm{d}s \\
&+ \int_0^t \mathrm{e}^{(t-s)A_2}(\nabla^{(2)}_{\sigma_s\mathrm{d}W_s}u_s^\lambda)(Z_s) + \int_0^t \mathrm{e}^{(t-s)A_2}(\lambda u_s^\lambda(Z_s) - b_s(Z_s))\mathrm{d}s,
\end{aligned}
$$

which then gives (4.2) by combining the second equation in (4.1). □

As a consequence of Theorem 4.1, we have the following pathwise uniqueness result.

Corollary 4.2. *Assume* **(H1)-(H4)** *and let* $b : [0,\infty) \to C_b(\mathbb{H};\mathbb{H}_2)$ *be measurable such that*

$$\sup_{t\in[0,T]} \|b_t\|_{\gamma_\alpha,\phi} < \infty, \quad T > 0$$

holds for some $\alpha \in (\frac{2}{3}, 1)$ *and* $\phi \in \mathscr{D}_1$. *Let* $(X_t, Y_t)_{t\geq 0}$ *and* $(\tilde{X}_t, \tilde{Y}_t)_{t\geq 0}$ *be two adapted continuous processes on* $\mathbb{H}$ *with* $(X_0, Y_0) = (\tilde{X}_0, \tilde{Y}_0)$. *For any* $n \geq 1$ *let*

$$\tau_n = n \wedge \inf\{t \geq 0 : |(X_t, Y_t)| \geq n\}, \quad \tilde{\tau}_n = n \wedge \inf\{t \geq 0 : |(\tilde{X}_t, \tilde{Y}_t)| \geq n\}.$$

If (X_t, Y_t) *and* $(\tilde{X}_t, \tilde{Y}_t)$ *are mild solutions to* (1.1) *for* $t \in [0, \tau_n \wedge \tilde{\tau}]$, *then* $\tau_n = \tilde{\tau}_n$ *and* $(X_t, Y_t) = (\tilde{X}_t, \tilde{Y}_t)$ *for all* $t \in [0, \tau_n]$.

When $\mathbb{H}_1, \mathbb{H}_2$ *are finite-dimensional, the assertion holds for* $\mathscr{D}_2$ *in place of* $\mathscr{D}_1$.

Proof. Due to Theorem 4.1 and Proposition 3.1, the proof is similar to that of [11, Proposition 3.1]. We address it here for complement. Write

$$T_n := \tau_n \wedge \tilde{\tau}_n,\ Z_t := (X_t, Y_t),\ \tilde{Z}_t := (\tilde{X}_t, \tilde{Y}_t).$$

It suffices to prove that for any $T > 0$,

$$\int_0^T \mathbb{E}\big[1_{\{s<T_n\}}|Z_s - \tilde{Z}_s|^2\big]\mathrm{d}s = 0. \tag{4.14}$$

By Theorem 4.1 for $\tau = T_n$, we have

$$
\begin{aligned}
h(t) &:= \mathbb{E}\big[1_{\{t<T_n\}}|Z_t - \tilde{Z}_t|^2\big] = \mathbb{E}\big[1_{\{t<T_n\}}\big(|X_t - \tilde{X}_t|^2 + |Y_t - \tilde{Y}_t|^2\big)\big] \\
&\leq C_1\bigg[\mathbb{E}\int_0^t 1_{\{s<T_n\}}|Y_s - \tilde{Y}_s|^2\mathrm{d}s + \lambda\mathbb{E}\int_0^t 1_{\{s<T_n\}}|u_s^\lambda(Z_s) - u_s^\lambda(\tilde{Z}_s)|^2\mathrm{d}s \\
&\qquad + I(t) + J(t)\bigg] + 2\mathbb{E}\big[1_{\{t<T_n\}}|u_t^\lambda(Z_t) - u_{t\wedge T_n}^\lambda(\tilde{Z}_{t\wedge T_n})|^2\big]
\end{aligned}
$$

for some constant $C_1 > 0$ and $t \in [0, T]$, where

$$
\begin{aligned}
I(t) &:= \mathbb{E}\bigg|1_{\{t<T_n\}}\int_0^t A_2\mathrm{e}^{(t-s)A_2}\big(u_s^\lambda(Z_s) - u_s^\lambda(\tilde{Z}_s)\big)\mathrm{d}s\bigg|^2, \\
J(t) &:= \mathbb{E}\int_0^t 1_{\{s<T_n\}}\|\mathrm{e}^{(t-s)A_2}\|_{HS}^2\big\|\nabla^{(2)}u_s^\lambda(Z_s) - \nabla^{(2)}u_s^\lambda(\tilde{Z}_s)\big\|^2\mathrm{d}s.
\end{aligned} \tag{4.15}
$$

B (3.3), we have

$$|u_s^\lambda(Z_s)-u_s^\lambda(\tilde Z_s)|+\|\nabla^{(2)}u_s^\lambda(Z_s)-\nabla^{(2)}u_s^\lambda(\tilde Z_s)\|\le\theta(\lambda)|Z_s-\tilde Z_s|. \tag{4.16}$$

This implies that for all $r\in[0,T]$,

$$g(r):=\int_0^r h(t)\mathrm{d}t\le C_2\int_0^r g(t)\mathrm{d}t+C_2\int_0^r\big(I(t)+J(t)\big)\mathrm{d}t+2\theta(\lambda)^2g(r). \tag{4.17}$$

By (4.15) and Hölder's inequality, we have

$$\begin{aligned}I(t)&=\sum_{i\ge1}\mathbb{E}\Big|1_{\{t<T_n\}}\int_0^t\lambda_i\mathrm{e}^{-(t-s)\lambda_i}\langle u_s^\lambda(Z_s)-u_s^\lambda(\tilde Z_s),e_i\rangle\mathrm{d}s\Big|^2\\&\le\sum_{i\ge1}\mathbb{E}\int_0^t 1_{\{s<T_n\}}\lambda_i\mathrm{e}^{-(t-s)\lambda_i}\langle u_s^\lambda(Z_s)-u_s^\lambda(\tilde Z_s),e_i\rangle^2\mathrm{d}s,\end{aligned}$$

which, together with Fubini's theorem and (4.16), yields

$$\begin{aligned}\mathbb{E}\int_0^r I(t)\mathrm{d}t&\le\sum_{i\ge1}\lambda_i\mathbb{E}\int_0^r\mathrm{d}t\int_0^t 1_{\{s<T_n\}}\mathrm{e}^{-(t-s)\lambda_i}\langle u_s^\lambda(Z_s)-u_s^\lambda(\tilde Z_s),e_i\rangle^2\mathrm{d}s\\&=\sum_{i=1}^\infty\mathbb{E}\int_0^r 1_{\{s<T_n\}}\langle u_s^\lambda(Z_s)-u_s^\lambda(\tilde Z_s),e_i\rangle^2\mathrm{d}s\int_s^r\lambda_i\mathrm{e}^{-(t-s)\lambda_i}\mathrm{d}t\\&\le\mathbb{E}\int_0^r 1_{\{s<T_n\}}|u_s^\lambda(Z_s)-u_s^\lambda(\tilde Z_s)|^2\mathrm{d}s\\&\le\theta(\lambda)^2\int_0^r\mathbb{E}\big[1_{\{s<T_n\}}|Z_s-\tilde Z_s|^2\big]\mathrm{d}s=\theta(\lambda)^2g(r).\end{aligned} \tag{4.18}$$

On the other hand, by (4.15), (4.16) and (2.15), we have

$$\begin{aligned}\mathbb{E}\int_0^r J(t)\mathrm{d}t&=\mathbb{E}\int_0^r 1_{\{s<T_n\}}\|\nabla^{(2)}u_s^\lambda(Z_s)-\nabla^{(2)}u_s^\lambda(\tilde Z_s)\|^2\mathrm{d}s\int_s^r\|\mathrm{e}^{(t-s)A_2}\|_{HS}^2\mathrm{d}t\\&\le c_2\theta(\lambda)^2\int_0^r h(s)\mathrm{d}s=c_2\theta(\lambda)^2g(r).\end{aligned}$$

Substituting this and (4.18) into (4.17), we arrive at

$$g(r)\le C_2\int_0^r g(t)\mathrm{d}t+(C_2+2+c_2)\theta(\lambda)^2g(r),\quad r\in[0,T].$$

Therefore, by letting λ be large enough so that $(C_2+2+c_2)\theta(\lambda)^2<\frac12$, then using Gronwall's inequality, we obtain $g(T)=0$. The proof of the first assertion is complete.

Finally, when $\mathbb{H}_1$ and $\mathbb{H}_2$ are finite-dimensional and the condition on b holds for $\phi\in\mathscr{D}_2$, in the above proof, we only need to give a different treatment for $\mathbb{E}\int_0^r J(t)\mathrm{d}t$. Notice that

in this case, $\sup_{t\ge 0}\|\mathrm{e}^{tA_2}\|_{HS}<\infty$, and by taking $r=1\wedge\sqrt{g(t)}$ in (3.4), we obtain

$$\begin{aligned}&\|\nabla^{(2)}u_s(Z_s)-\nabla^{(2)}u_s(\tilde Z_s)\|\\ &\le C_3\bigg\{\sqrt{g(t)}+|Z_s-\tilde Z_s|\bigg(1+\int_{g(t)^{\delta/2}\wedge 1}^1\frac{\phi(s')}{s'}\mathrm{d}s'\bigg)\bigg\},\ \ s\in[0,T],t>s\end{aligned}\tag{4.19}$$

for some constant $C_3>0$. Hence, for some constants $C_4,C_5>0$, we have

$$\begin{aligned}\mathbb{E}\int_0^r J(t)\mathrm{d}t&=\mathbb{E}\int_0^r\mathrm{d}t\int_0^t 1_{\{s<T_n\}}\|\mathrm{e}^{(t-s)A_2}\|_{HS}^2\big\|\nabla^{(2)}u_s^\lambda(Z_s)-\nabla^{(2)}u_s^\lambda(\tilde Z_s)\big\|^2\mathrm{d}s\\ &\le C_4\int_0^r\mathrm{d}t\int_0^t\bigg\{g(t)+h(s)\bigg(1+\int_{g(t)^{\delta/2}\wedge 1}^1\frac{\phi(s')}{s'}\mathrm{d}s'\bigg)^2\bigg\}\mathrm{d}s\\ &\le C_5\int_0^r g(t)\bigg(1+\int_{g(t)^{\delta/2}\wedge 1}^1\frac{\phi(s)}{s}\mathrm{d}s\bigg)^2\mathrm{d}t,\quad r\in[0,T].\end{aligned}$$

Substituting this and (4.18) into (4.17) for large λ, we arrive at

$$g(r)\le C_6\int_0^r g(t)\bigg(1+\int_{g(t)^{\delta/2}\wedge 1}^1\frac{\phi(s)}{s}\mathrm{d}s\bigg)^2\mathrm{d}t=\int_0^r\rho(g(t))\mathrm{d}t,\quad r\in[0,T]$$

for some constant $C_6>0$, where $\rho(t):=C_6t\left(1+\int_{t^{\delta/2}\wedge 1}^1\frac{\phi(s)}{s}\mathrm{d}s\right)^2$. Since $\phi\in\mathscr{D}_2$, we have

$$\int_0^1\frac{\mathrm{d}t}{\rho(t)}=\frac{2}{C_6\delta}\int_0^1\frac{\mathrm{d}t}{t\big(1+\int_{t\wedge 1}^1\frac{\phi(s)}{s}\mathrm{d}s\big)^2}=\infty.$$

Therefore, by Bihari's inequality, this implies $g(T)=0$ which is equivalent to (4.14). $\square$

5 Proofs of Theorem 1.1 and Theorem 1.2

Due to Corollary 4.2, the proofs are more or less similar to [11, §5].

Proof of Theorem 1.1. We split the proof into the following three steps.

(a) Firstly, we assume that for any $T>0$ there exist $\alpha\in(\frac{2}{3},1)$ and $\phi\in\mathscr{D}_1$ such that $\sup_{t\in[0,T]}\|b_t\|_{\gamma_\alpha,\phi}<\infty$. By Girsanov's theorem, we see that for any $T>0$ and any initial value (X_0,Y_0), the equation (1.1) has a weak mild solution up to time T. Moreover, by Corollary 4.2, the mild solution is pathwise unique up to arbitrary time $T>0$. So, by the Yamada-Watanabe principle [12] (see [8, Theorem 2] or [10] for the result in infinite dimensions), for any initial value, the equation (1.1) has a unique mild solution which is non-explosive (i.e. the life time $\zeta=\infty$).

(b) In general, take $\psi\in C_b^\infty([0,\infty))$ such that $0\le\psi\le 1,\psi(r)=1$ for $r\in[0,1]$ and $\psi(r)=0$ for $r\ge 2$. For any $m\ge 1$, let

$$b_t^{[m]}(z)=b_{t\wedge m}(z)\psi(|z|/m),\quad t\ge 0,z\in\mathbb{H}.$$

Then by the condition in Theorem 1.1(1), for any $m \geq 1$ and $T > 0$, there exist $\alpha \in (\frac{2}{3}, 1)$ and $\phi \in \mathscr{D}_1$ such that

$$\sup_{t \in [0,T]} \|b_t^{(m)}\|_{\gamma_\alpha, \phi} < \infty.$$

So, for any initial value (X_0, Y_0), the equation (1.1) for $b^{[m]}$ in place of b has a unique mild solution $(X_t^{(m)}, Y_t^{(m)})_{t \geq 0}$ which is non-explosive. Let

$$\tau_m = m \wedge \inf\big\{t \geq 0 : |(X_t^{(m)}, Y_t^{(m)})| \geq m\big\}, \quad m \geq 1.$$

Since $b_s^{[m]}(z) = b_s(z)$ for $s \leq m$ and $|z| \leq m$, by Corollary 4.2, for any $n, m \geq 1$ we have $(X_t^{(n)}, Y_t^{(n)}) = (X_t^{(m)}, Y_t^{(m)})$ for $t \in [0, \tau_n \wedge \tau_m]$. In particular, τ_m is increasing in m. Let $\zeta = \lim_{m \to \infty} \tau_m$ and

$$(X_t, Y_t) = \sum_{m=1}^{\infty} 1_{[\tau_{m-1}, \tau_m)}(t)(X_t^{(m)}, Y_t^{(m)}) \quad \tau_0 := 0, t \in [0, \zeta).$$

Then it is easy to see that $(X_t, Y_t)_{t \in [0,\zeta)}$ is a mild solution to (1.1) with life time ζ and, due to Corollary 4.2, the mild solution is unique. Thus, the assertion (1) is proved.

(c) Let (1.2) hold for some positive increasing ℓ, h such that $\int_1^\infty \frac{\mathrm{d}s}{\ell_t(s)} = \infty, t \geq 0$. Let $(X_t, Y_t)_{t \in [0,\zeta)}$ be a mild solution to (1.1) with life time ζ. Let $\xi_t = \int_0^t \mathrm{e}^{(t-s)A_2} \sigma_s \mathrm{d}W_s$. Due to the boundedness of σ and **(H3)**, ξ_t is an adapted continuous process on $\mathbb{H}_2$ up to the life time ζ with

$$\eta_T = |Y_0|^2 + 2 \int_0^{T \wedge \zeta} h_{T \wedge \zeta}(|\xi_s|) \mathrm{d}s < \infty, \quad T > 0.$$

Then $\tilde{Y}_t := Y_t - \xi_t$ solves the equation

$$\mathrm{d}\tilde{Y}_t = \big(A_2 \tilde{Y}_t + b_t(X_t, \tilde{Y}_t + \xi_t)\big)\mathrm{d}t, \quad \tilde{Y}_0 = \tilde{Y}_0, \quad t < \zeta.$$

Due to (1.2), the increasing property of h, ℓ, and $A_2 \leq 0$, this implies that for any $T > 0$,

$$\begin{aligned} \mathrm{d}|\tilde{Y}_t|^2 &\leq 2\langle \tilde{Y}_t, b_t(X_t, \tilde{Y}_t + \xi_t)\rangle \,\mathrm{d}t \\ &\leq 2\big(\ell_{T \wedge \zeta}(|X_t|^2 + |\tilde{Y}_t|^2) + h_T(|\xi_t|)\big)\mathrm{d}t, \quad |\tilde{Y}_0|^2 = |Y_0|^2, t < \zeta \wedge T. \end{aligned}$$

So,

$$|\tilde{Y}_t|^2 \leq \eta_T + 2 \int_0^t \ell_T(|X_r|^2 + |\tilde{Y}_r|^2) \mathrm{d}r, \quad T > 0, t \in [0, T \wedge \zeta).$$

On the other hand, there exists a random variable $C > 1$ such that

$$|X_r|^2 = \left|\mathrm{e}^{rA_1} X_0 + \int_0^r \mathrm{e}^{(r-s)A_1} B \tilde{Y}_s \mathrm{d}s\right|^2 \leq C + (C-1) \sup_{s \in [0,r]} |\tilde{Y}_s|^2, \quad r \in [0, T \wedge \zeta). \tag{5.1}$$

Therefore, $g(t) := \sup_{s \in [0,t]} |\tilde{Y}_s|^2$ satisfies

$$g(t) \leq \eta_T + 2 \int_0^t \ell_T(C + Cg(r)) \mathrm{d}r, \quad T > 0, t \in [0, T \wedge \zeta).$$

Letting

$$\Gamma_T(s) = \int_1^s \frac{\mathrm{d}r}{2\ell_{T\wedge\zeta}(C+Cr)},$$

by Biharis's inequality, we obtain

$$g(t) \le \Gamma_T^{-1}\big(\Gamma_T(\eta_T)+t\big), \quad T>0, t\in[0,\zeta\wedge T). \tag{5.2}$$

This implies $\mathbb{P}(\zeta<\infty)=0$, i.e. (X_t,Y_t) is non-explosive. Indeed, by the property of the life time and (5.1) and (5.2), on the set $\{\zeta\le T\}$ we have $\mathbb{P}$-a.s.

$$\infty = \limsup_{t\uparrow\zeta}(|X_t|^2+|Y_t|^2) \le \limsup_{t\uparrow\zeta} g(t) \le \Gamma_T^{-1}\big(\Gamma_T(\eta_T)+T\big) < \infty,$$

where the last step is due to the fact that $\Gamma_T(r)\uparrow\infty$ as $r\uparrow\infty$, which implies $\Gamma_T^{-1}(r)<\infty$ for any $r\in(0,\infty)$. This contradiction means that $\mathbb{P}(\zeta\le T)=0$ holds for all $T\in(0,\infty)$. Hence, $\mathbb{P}(\zeta<\infty)=0$. □

Proof of Theorem 1.2. Since $\mathbb{H}$ is finite-dimensional, we may apply Itô's formula for $|X_t|^2$ directly to prove the non-explosion from (1.7) as in (c) in the proof of Theorem 1.1. So, as explained in the proof of Theorem 1.1, it remains to prove the pathwise uniqueness for b satisfying

$$\sup_{t\in[0,T]} \|b_t\|_{\gamma_\alpha,\phi} < \infty, \quad T>0,$$

where $\alpha\in(\frac{2}{3},1)$ and $\phi\in\mathscr{D}_2$. To apply Corollary 2.4, we reformulate (1.6) as (1.1) for $A_1=A, A_2=I_d$, and b_t-I_d in place of b_t, where I_d is the identity operator on $\mathbb{R}^d$. Then assumptions **(H1)**, **(H3)** and **(H4)** are trivial since $\mathbb{H}$ is finite-dimensional. Moreover, **(H2)** holds for $A_0=I_m-A$. Thus, the pathwise uniqueness follows from Corollary 2.4. □

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